

# LAGUERRE FUNCTIONS AND THEIR APPLICATIONS TO TEMPERED FRACTIONAL DIFFERENTIAL EQUATIONS ON INFINITE INTERVALS

SHENG CHEN<sup>1</sup>, JIE SHEN<sup>1,2</sup> AND LI-LIAN WANG<sup>3</sup>

**ABSTRACT.** Tempered fractional derivatives originated from the tempered fractional diffusion equations (**TFDEs**) modeled on the whole space  $\mathbb{R}$  (see [23]). For numerically solving TFDEs, two kinds of generalized Laguerre functions were defined and some important properties were proposed to establish the approximate theory. The related prototype tempered fractional differential problems was proposed and solved as the guidance. TFDEs are numerically solved by two domains Laguerre spectral method and the numerical experiments show some properties of the TFDEs and verify the efficiency of the spectral scheme.

## 1. INTRODUCTION

The normal diffusion equation  $\partial_t p(x, t) = \partial_x^2 p(x, t)$  can be derived from the Brownian motion which describes the particle's random walks. Over the last few decades, a large body of literature has demonstrated that anomalous diffusion, in which the mean square variance grows faster (super-diffusion) or slower (sub-diffusion) than in a Gaussian process, offers a superior fit to experimental data observed in many important practical applications, e.g., in physical science [14, 17, 18, 19], finance [11, 16, 25], biology [13, 5] and hydrology [4, 8, 9]. The anomalous diffusion equation takes the form

$$\partial_t^\nu p(x, t) = \partial_x^\mu p(x, t), \quad (1.1)$$

where  $0 < \nu \leq 1$  and  $0 < \mu < 2$  (cf. [17] for a review on this subject), whose solution exhibits heavy tails, i.e., power law decays at infinity. In order to "temper" the power law decay, the authors of [23] applied an exponential factor  $e^{-\lambda|x|}$  to the particle jump density, and showed that the Fourier transform of the tempered probability density function  $p(x, t)$  takes the form

$$\mathcal{F}[p](\omega, t) = e^{-[pA_+^{\mu, \lambda}(\omega) + qA_-^{\mu, \lambda}(\omega)]Dt}, \quad 0 < \mu < 2,$$

where  $0 \leq p \leq 1$ ,  $q = 1 - p$ ,  $D$  is a constant and

$$A_\pm^{\mu, \lambda}(\omega) := \begin{cases} (\lambda \pm i\omega)^\mu - \lambda^\mu, & 0 < \mu < 1, \\ (\lambda \pm i\omega)^\mu - \lambda^\mu - \pm i\omega\mu\lambda^{\mu-1}, & 1 < \mu < 2. \end{cases}$$

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<sup>1</sup>Fujian Provincial Key Laboratory on Mathematical Modeling & High Performance Scientific Computing and School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, P. R. China. This work is supported in part by NSFC grants 11371298, 11421110001, 91630204 and 51661135011.

<sup>2</sup>Department of Mathematics, Purdue University, West Lafayette, IN 47907-1957, USA. J.S. is partially supported by NSF grant DMS-1620262 and AFOSR grant FA9550-16-1-0102.

<sup>3</sup>Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371, Singapore. L.W. is partially supported by Singapore MOE AcRF Tier 1 Grant (RG 15/12), and Singapore MOE AcRF Tier 2 Grant (MOE 2013-T2-1-095, ARC 44/13).

Moreover, they defined tempered fractional derivative operators  $\partial_{\pm, x}^{\mu, \lambda}$  through Fourier transform:  $\mathcal{F}[\partial_{\pm, x}^{\mu, \lambda} u](\omega) = A_{\pm}^{\mu, \lambda}(\omega) \mathcal{F}[u](\omega)$ , and derived the tempered fractional diffusion equation (**TFDE**):

$$\partial_t u(x, t) = (-1)^k C_T \{p \partial_{+, x}^{\mu, \lambda} + q \partial_{-, x}^{\mu, \lambda}\} u(x, t), \quad \mu \in (k-1, k), \quad k = 1, 2. \quad (1.2)$$

It has been argued that tempered anomalous diffusion models have advantages over the normal dissuasion models in some applications of geophysics [15, 31] and finance [6].

It is a challenging task to numerically solve the tempered fractional diffusion equation (1.2), due particularly to (i) the non-local nature of tempered fractional derivatives; and (ii) the unboundedness of the domain. In [23], the authors used a finite-difference approach on a truncated domain. In [29], the authors considered tempered derivatives on a finite interval and derived an efficient Petrov-Galerkin method for solving tempered fractional ODEs by using the eigenfunctions of tempered fractional Sturm-Liouville problems. In [12], the authors used Laguerre functions to approximate the substantial fractional ODEs, which are similar to those we consider in Section 3, on the half line. In order to avoid the difficulty of assigning boundary conditions at the truncated boundary, we shall deal with the unbounded domain directly in this paper.

Since the tempered fractional diffusion equation is derived from the random walk on the whole line, one is tempted to use Hermite polynomials/functions which are suitable for many problems on the whole line [26]. Unfortunately, due to the exponential factor  $e^{\lambda|x|}$  in the tempered fractional derivatives, Hermite polynomials/functions are not suitable basis functions. Instead, as we will show in Section 3, properly defined generalized Laguerre functions (**GLFs**) enjoy particularly simple form under the action of tempered fractional derivatives, just as the relations between generalized Jacobi functions and usual fractional derivatives [7]. Hence, the main goal of this paper is to design efficient spectral methods using **GLFs** to solve the tempered fractional diffusion equation (1.2) in various situations. However, Laguerre polynomials/functions are mutually orthogonal on the half line, how do we use them to deal with (1.2) on the whole line? We shall first consider special cases of (1.2) with  $p = 1, q = 0$  or  $p = 0, q = 1$ . In these cases, we can reduce (1.2) to the half line, and the **GLFs** can be naturally used. For the general case, we shall employ a two-domain spectral-element method, and use **GLFs** as basis functions on each subdomain.

The rest of the paper is organized as follows. In the next section, we present the definition of the tempered fractional derivatives, and recall some useful properties of Laguerre polynomials. In Sections 3, we define two classes of generalized Laguerre functions, study their approximation properties, and apply them for solving simple one sided tempered fractional equations. In Section 4, we develop a spectral-Galerkin method for solving a tempered fractional diffusion equation on the half line. Finally, we present a spectral-Galerkin method for solving the tempered fractional diffusion equation on the whole line in Section 5. Some concluding remarks are given in the last section.

## 2. PRELIMINARIES

Let  $\mathbb{N}$  and  $\mathbb{R}$  be respectively the sets of positive integers and real numbers. We further denote

$$\mathbb{N}_0 := \{0\} \cup \mathbb{N}, \quad \mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}, \quad \mathbb{R}^- := \{x \in \mathbb{R} : x < 0\}, \quad \mathbb{R}_0^\pm := \mathbb{R}^\pm \cup \{0\}. \quad (2.1)$$

**2.1. Usual (non-tempered) fractional integrals and derivatives.** Recall the definitions of the fractional integrals and fractional derivatives in the sense of Riemann-Liouville (see e.g., [21]).

**Definition 2.1 (Riemann-Liouville fractional integrals and derivatives).** For  $a, b \in \mathbb{R}$  or  $a = -\infty, b = \infty$ , and  $\mu \in \mathbb{R}^+$ , the left and right fractional integrals are respectively defined as

$${}_a I_x^\mu u(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{u(y)}{(x-y)^{1-\mu}} dy, \quad {}_x I_b^\mu u(x) = \frac{1}{\Gamma(\mu)} \int_x^b \frac{u(y)}{(y-x)^{1-\mu}} dy, \quad x \in \Lambda := (a, b). \quad (2.2)$$

For real  $s \in [k-1, k)$  with  $k \in \mathbb{N}$ , the left-sided Riemann-Liouville fractional derivative (LRLFD) of order  $s$  is defined by

$${}_a D_x^s u(x) = \frac{1}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_a^x \frac{u(y)}{(x-y)^{s-k+1}} dy, \quad x \in \Lambda, \quad (2.3)$$

and the right-sided Riemann-Liouville fractional derivative (RRLFD) of order  $s$  is defined by

$${}_x D_b^s u(x) = \frac{(-1)^k}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_x^b \frac{u(y)}{(y-x)^{s-k+1}} dy, \quad x \in \Lambda. \quad (2.4)$$

From the above definitions, it is clear that for any  $k \in \mathbb{N}_0$ ,

$${}_x D_b^k = (-1)^k D^k, \quad {}_a D_x^k = D^k, \quad \text{where } D^k := \frac{d^k}{dx^k}. \quad (2.5)$$

Therefore, we can express the RLFD as

$${}_a D_x^s u(x) = D^k \{ {}_a I_x^{k-s} u(x) \}; \quad {}_x D_b^s u(x) = (-1)^k D^k \{ {}_x I_b^{k-s} u(x) \}. \quad (2.6)$$

According to [10, Thm. 2.14], we have that for any finite  $a$  and any  $v \in L^1(\Lambda)$ , and real  $s \geq 0$ ,

$${}_a D_x^s {}_a I_x^s f(x) = f(x), \quad \text{a.e. in } \Lambda. \quad (2.7)$$

Note that by commuting the integral and derivative operators in (2.6), we define the Caputo fractional derivatives:

$${}_a^C D_x^s u(x) = {}_a I_x^{k-s} \{ D^k u(x) \}; \quad {}_x^C D_b^s u(x) = (-1)^k {}_x I_b^{k-s} \{ D^k u(x) \}. \quad (2.8)$$

For an affine transform  $x = \lambda t$ ,  $\lambda > 0$ , on account of

$$\begin{aligned} {}_a I_t^\mu v(\lambda t) &= \frac{1}{\Gamma(\mu)} \int_a^t \frac{v(\lambda s)}{(t-s)^{1-\mu}} ds = \frac{\lambda^{-\mu}}{\Gamma(\mu)} \int_a^t \frac{v(\lambda s)}{(\lambda t - \lambda s)^{1-\mu}} \lambda ds \\ &= \frac{\lambda^{-\mu}}{\Gamma(\mu)} \int_{\lambda a}^x \frac{v(y)}{(x-y)^{1-\mu}} dy = \lambda^{-\mu} {}_{\lambda a} I_x^\mu v(x), \end{aligned}$$

and  $\frac{d}{dt} = \lambda \frac{d}{dx}$ , we derive from Definition 2.1 that

$${}_a I_t^\mu v(\lambda t) = \lambda^{-\mu} {}_{\lambda a} I_x^\mu v(x), \quad {}_a D_t^s v(\lambda t) = \lambda^s {}_{\lambda a} D_x^s v(x), \quad s, \mu, \lambda > 0. \quad (2.9)$$

Similarly, we have the following identities for the right fractional derivative:

$${}_t I_b^\mu v(\lambda t) = \lambda^{-\mu} {}_t I_{\lambda b}^\mu v(x), \quad {}_t D_b^s v(\lambda t) = \lambda^s {}_x D_{\lambda b}^s v(x), \quad s, \mu, \lambda > 0. \quad (2.10)$$

**2.2. Tempered fractional integrals and derivatives on  $\mathbb{R}$ .** Recently, Sabzikar et al. [23, (19)-(23)] introduced the tempered fractional integrals and derivatives on the whole line.

**Definition 2.2 (Tempered fractional integrals).** For  $\lambda \in \mathbb{R}_0^+$ , the left tempered fractional integral of a suitable function  $u(x)$  of order  $\mu \in \mathbb{R}^+$  is defined by

$$-_\infty I_x^{\mu, \lambda} u(x) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x \frac{e^{-\lambda(x-y)}}{(x-y)^{1-\mu}} u(y) dy, \quad x \in \mathbb{R}, \quad (2.11)$$

and the right tempered fractional integral of order  $\mu \in \mathbb{R}^+$  is defined by

$${}_x I_\infty^{\mu, \lambda} u(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty \frac{e^{-\lambda(y-x)}}{(y-x)^{1-\mu}} u(y) dy, \quad x \in \mathbb{R}. \quad (2.12)$$

It is evident that by (2.2) and (2.11)-(2.12), we have

$$-_\infty I_x^\mu = -_\infty I_x^{\mu, 0}, \quad {}_x I_\infty^\mu = {}_x I_\infty^{\mu, 0}, \quad (2.13)$$

and

$$-_\infty I_x^{\mu, \lambda} u(x) = e^{-\lambda x} -_\infty I_x^\mu \{ e^{\lambda x} u(x) \}, \quad {}_x I_\infty^{\mu, \lambda} u(x) = e^{\lambda x} {}_x I_\infty^\mu \{ e^{-\lambda x} u(x) \}. \quad (2.14)$$

As shown in [23], the tempered fractional derivative can be characterized by its Fourier transform. Recall that, for any  $u \in L^2(\mathbb{R})$ , its Fourier transform and inverse Fourier transform are defined by

$$\mathcal{F}[u](\omega) = \int_{-\infty}^{\infty} u(x) e^{-i\omega x} dx; \quad u(x) = \mathcal{F}^{-1}[\mathcal{F}[u](\omega)](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[u](\omega) e^{i\omega x} d\omega. \quad (2.15)$$

There holds the well-known Parseval's identity:

$$\int_{-\infty}^{\infty} u(x) \bar{v}(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[u](\omega) \overline{\mathcal{F}[v](\omega)} d\omega, \quad (2.16)$$

where  $\bar{v}$  is the complex conjugate of  $v$ . Let  $H(x)$  be the Heaviside function, i.e.,  $H(x) = 1$  for  $x \geq 0$ , and vanishing for all  $x < 0$ . Then we can reformulate the left tempered fractional integral as

$$\begin{aligned} {}_{-\infty}I_x^{\mu, \lambda} u(x) &= \frac{1}{\Gamma(\mu)} \int_0^{\infty} y^{\mu-1} e^{-\lambda y} u(x-y) dy = \frac{1}{\Gamma(\mu)} \int_{-\infty}^{\infty} y^{\mu-1} e^{-\lambda y} H(y) u(x-y) dy \\ &= (K * u)(x), \quad \text{where } K(x) := x^{\mu-1} e^{-\lambda x} H(x) / \Gamma(\mu). \end{aligned} \quad (2.17)$$

Note that  $K(x)$  is related to the particle jump density (cf. [23, (8)]). Using the formula:  $\mathcal{F}[K](\omega) = (\lambda + i\omega)^{-\mu}$ , and the convolution property of Fourier transform (see, e.g., [24, 27]), we derive

$$\mathcal{F}[{}_{-\infty}I_x^{\mu, \lambda} u](\omega) = \mathcal{F}[K * u](\omega) = \mathcal{F}[K](\omega) \mathcal{F}[u](\omega) = (\lambda + i\omega)^{-\mu} \mathcal{F}[u](\omega). \quad (2.18)$$

Similarly, the Fourier transform of the right tempered fractional integral is

$$\mathcal{F}[{}_xI_{\infty}^{\mu, \lambda} u](\omega) = (\lambda - i\omega)^{-\mu} \mathcal{F}[u](\omega). \quad (2.19)$$

In view of (2.18)-(2.19), Sabzikar et al. [23] then introduced the left and right tempered fractional derivatives as follows.

**Definition 2.3 (Tempered fractional derivatives).** For  $\lambda \in \mathbb{R}_0^+$ , the left and right tempered fractional derivatives of order  $\mu \in \mathbb{R}^+$  of a suitable function  $u(x)$ , are defined by

$$\mathcal{F}[{}_{-\infty}D_x^{\mu, \lambda} u](\omega) = (\lambda + i\omega)^{\mu} \mathcal{F}[u](\omega), \quad \mathcal{F}[{}_xD_{\infty}^{\mu, \lambda} u](\omega) = (\lambda - i\omega)^{\mu} \mathcal{F}[u](\omega), \quad (2.20)$$

that is, for any  $x \in \mathbb{R}$ ,

$${}_{-\infty}D_x^{\mu, \lambda} u(x) = \mathcal{F}^{-1}[(\lambda + i\omega)^{\mu} \mathcal{F}[u](\omega)](x), \quad {}_xD_{\infty}^{\mu, \lambda} u(x) = \mathcal{F}^{-1}[(\lambda - i\omega)^{\mu} \mathcal{F}[u](\omega)](x). \quad (2.21)$$

Introduce the space

$$W_{\lambda}^{\mu, 2}(\mathbb{R}) := \left\{ u \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (\lambda^2 + \omega^2)^{\mu} |\mathcal{F}[u](\omega)|^2 d\omega < \infty \right\}, \quad \mu, \lambda \in \mathbb{R}^+. \quad (2.22)$$

Thanks to the Parseval's identity (2.16), the above tempered fractional derivatives are well-defined for any  $u \in W_{\lambda}^{\mu, 2}(\mathbb{R})$ . Moreover, one verifies from (2.18)-(2.21) that

$$\begin{aligned} {}_{-\infty}I_x^{\mu, \lambda} {}_{-\infty}D_x^{\mu, \lambda} u(x) &= u(x), \quad {}_xI_{\infty}^{\mu, \lambda} {}_xD_{\infty}^{\mu, \lambda} u(x) = u(x), \quad \forall u \in W_{\lambda}^{\mu, 2}(\mathbb{R}); \\ {}_{-\infty}D_x^{\mu, \lambda} {}_{-\infty}I_x^{\mu, \lambda} u(x) &= u(x), \quad {}_xD_{\infty}^{\mu, \lambda} {}_xI_{\infty}^{\mu, \lambda} u(x) = u(x), \quad \forall u \in L^2(\mathbb{R}). \end{aligned} \quad (2.23)$$

Similar to (2.14), we have the following explicit representations.

**Proposition 2.1.** For any  $u \in W_{\lambda}^{\mu, 2}(\mathbb{R})$ , with  $\lambda \in \mathbb{R}_0^+$ , the left and right tempered fractional derivatives of order  $\mu \in [k-1, k)$  with  $k \in \mathbb{N}$ , have the explicit representations:

$${}_{-\infty}D_x^{\mu, \lambda} u(x) = e^{-\lambda x} {}_{-\infty}D_x^{\mu} \{e^{\lambda x} u(x)\}, \quad {}_xD_{\infty}^{\mu, \lambda} u(x) = e^{\lambda x} {}_xD_{\infty}^{\mu} \{e^{-\lambda x} u(x)\}, \quad (2.24)$$

where  ${}_{-\infty}D_x^\mu$  and  ${}_xD_\infty^\mu$  are the Riemann-Liouville fractional derivative operators in Definition 2.1. Alternatively, we have

$$\begin{aligned} {}_{-\infty}D_x^{\mu,\lambda}u(x) &= (D + \lambda)^k \{ {}_{-\infty}I_x^{k-\mu,\lambda}u(x) \} = (D + \lambda)^k \{ e^{-\lambda x} {}_{-\infty}I_x^{k-\mu} \{ e^{\lambda x} u(x) \} \}; \\ {}_xD_\infty^{\mu,\lambda}u(x) &= (D - \lambda)^k \{ {}_xI_\infty^{k-\mu}u(x) \} = (D - \lambda)^k \{ e^{\lambda x} {}_xI_\infty^{k-\mu} \{ e^{-\lambda x} u(x) \} \}. \end{aligned} \quad (2.25)$$

*Proof.* Using the properties of Fourier transform:  $\mathcal{F}[e^{-\lambda x} D^k v] = (\lambda + i\omega)^k \mathcal{F}[e^{-\lambda x} v]$ , and

$$\begin{aligned} \mathcal{F}[e^{-\lambda x} {}_{-\infty}D_x^\mu \{ e^{\lambda x} u(x) \}](\omega) &= \mathcal{F}[e^{-\lambda x} D^k {}_{-\infty}I_x^{k-\mu} \{ e^{\lambda x} u(x) \}](\omega) \\ &= (\lambda + i\omega)^k \mathcal{F}[e^{-\lambda x} {}_{-\infty}I_x^{k-\mu} \{ e^{\lambda x} u(x) \}](\omega) \\ &= (\lambda + i\omega)^k \mathcal{F}[{}_{-\infty}I_x^{k-\mu,\lambda}u(x)](\omega) = (\lambda + i\omega)^\mu \mathcal{F}[u](\omega). \end{aligned} \quad (2.26)$$

This verifies (2.20)-(2.21). Similarly, we can derive the second representation of  ${}_xD_\infty^{\mu,\lambda}u$  in (2.24).

The alternative form (2.25) can be derived by induction. Here we only verify the left tempered derivative.

- For  $\mu \in [0, 1)$ , we derive from (2.14) and (2.24) that

$$\begin{aligned} {}_{-\infty}D_x^{\mu,\lambda}u(x) &= e^{-\lambda x} {}_{-\infty}D_x^\mu \{ e^{\lambda x} u(x) \} = e^{-\lambda x} D {}_{-\infty}I_x^{1-\mu} \{ e^{\lambda x} u(x) \} \\ &= e^{-\lambda x} D [e^{\lambda x} {}_{-\infty}I_x^{1-\mu,\lambda} \{ e^{\lambda x} u(x) \}] = (D + \lambda) \{ {}_{-\infty}I_x^{1-\mu,\lambda} \{ e^{\lambda x} u(x) \} \}, \end{aligned}$$

which verifies the identity with  $k = 1$ .

- For  $\mu \in [k - 2, k - 1)$ , we assume that (2.25) is true. We next verify the identity holds for  $\mu \in [k - 1, k)$ .

$$\begin{aligned} (D + \lambda)^k \{ {}_{-\infty}I_x^{k-\mu,\lambda}u(x) \} &= (D + \lambda)(D + \lambda)^{k-1} \{ e^{-\lambda x} {}_{-\infty}I_x^{k-\mu} \{ e^{\lambda x} u(x) \} \} \\ &= (D + \lambda) \{ e^{-\lambda x} {}_{-\infty}D_x^{\mu-1} \{ e^{\lambda x} u(x) \} \} = e^{-\lambda x} {}_{-\infty}D_x^\mu \{ e^{\lambda x} u(x) \} = {}_{-\infty}D_x^{\mu,\lambda}u(x). \end{aligned}$$

This ends the proof.  $\square$

We collect below some useful properties (cf. [23]).

**Lemma 2.1.** *Given  $\lambda > 0$  and  $\mu \in [k - 1, k)$ ,  $k \in \mathbb{N}$ , the tempered fractional derivative*

$${}_{-\infty}D_x^{\mu,\lambda}u(x) = {}_{-\infty}D_x^{k,\lambda} {}_{-\infty}I_x^{k-\mu,\lambda}u(x), \quad {}_xD_\infty^{\mu,\lambda}u(x) = {}_xD_\infty^{k,\lambda} {}_xI_\infty^{k-\mu,\lambda}u(x). \quad (2.27)$$

*In addition, we have*

$${}_{-\infty}I_x^{\mu+\nu,\lambda}u(x) = {}_{-\infty}I_x^{\mu,\lambda} {}_{-\infty}I_x^{\nu,\lambda}u(x), \quad {}_xI_\infty^{\mu+\nu,\lambda}u(x) = {}_xI_\infty^{\mu,\lambda} {}_xI_\infty^{\nu,\lambda}u(x), \quad (2.28)$$

$${}_{-\infty}D_x^{\mu+\nu,\lambda}u(x) = {}_{-\infty}D_x^{\mu,\lambda} {}_{-\infty}D_x^{\nu,\lambda}u(x), \quad {}_xD_\infty^{\mu+\nu,\lambda}u(x) = {}_xD_\infty^{\mu,\lambda} {}_xD_\infty^{\nu,\lambda}u(x), \quad (2.29)$$

$$({}_{-\infty}D_x^{\mu,\lambda}u, v) = (u, {}_xD_\infty^{\mu,\lambda}v), \quad ({}_xD_\infty^{\mu,\lambda}u, v) = (u, {}_{-\infty}D_x^{\mu,\lambda}v), \quad (2.30)$$

where  $\mu, \nu \geq 0$ .

**Remark 2.1.** For a suitable function  $f(x)$ ,  $x \in \mathbb{R}^+$ , its reflection  $g(y) = f(-y)$ ,  $y \in \mathbb{R}^-$  satisfies

$$\begin{aligned} {}_{-\infty}I_y^{\mu,\lambda}g(y) &= \frac{e^{-\lambda y}}{\Gamma(\mu)} \int_{-\infty}^y e^{\lambda \tau} (y - \tau)^{\mu-1} f(-\tau) d\tau \stackrel{t=-\tau}{=} \frac{e^{-\lambda y}}{\Gamma(\mu)} \int_{-y}^{\infty} e^{-\lambda t} (y + t)^{\mu-1} f(t) dt \\ &\stackrel{x=-y}{=} \frac{e^{\lambda x}}{\Gamma(\mu)} \int_x^{\infty} e^{-\lambda t} (t - x)^{\mu-1} f(t) dt = {}_xI_\infty^{\mu,\lambda}f(x). \end{aligned} \quad (2.31)$$

Hence, we can use (2.27) and derivative relation  $\frac{d^k}{dy^k} = (-1)^k \frac{d^k}{dx^k}$  to obtain the tempered derivative relation

$${}_{-\infty}D_y^{\mu,\lambda}f(-y) = {}_xD_\infty^{\mu,\lambda}f(x), \quad y = -x, \quad x \in \mathbb{R}^+. \quad (2.32)$$

$\square$

**2.3. Laguerre polynomials and some useful formulas.** For any  $a \in \mathbb{R}$  and  $j \in \mathbb{N}_0$ , we recall that the rising factorial in the Pochhammer symbol and the Gamma function have the relation:

$$(a)_0 = 1; \quad (a)_j := a(a+1) \cdots (a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)}, \quad \text{for } j \geq 1. \quad (2.33)$$

Recall the hypergeometric function (cf. [1]):

$${}_1F_1(a; b; x) = \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{x^j}{j!}, \quad a, b, x \in \mathbb{R}^+, \quad -b \notin \mathbb{N}_0. \quad (2.34)$$

If  $b - a > 0$ , then  ${}_1F_1(a; b; x)$  is absolutely convergent for all  $x \in \mathbb{R}$ . If  $a$  is a negative integer, then it reduces to a polynomial.

The Laguerre polynomial with parameter  $\alpha > -1$  is defined as in Szegő [28, (5.3.3)]:

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x), \quad n \geq 1, \quad x \in \mathbb{R}^+, \quad (2.35)$$

and  $L_0^{(\alpha)}(x) \equiv 1$ . Note that

$$L_n^{(\alpha)}(0) = \frac{(\alpha+1)_n}{n!}, \quad (2.36)$$

and the Laguerre polynomials (with  $\alpha > -1$ ) are orthogonal with respect to the weight function  $x^\alpha e^{-x}$ , namely,

$$\int_0^\infty L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) x^\alpha e^{-x} dx = \gamma_n^\alpha \delta_{mn}, \quad \gamma_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}. \quad (2.37)$$

They are eigenfunctions of the Sturm-Liouville problem:

$$x^{-\alpha} e^x \partial_x (x^{\alpha+1} e^{-x} \partial_x L_n^{(\alpha)}(x)) + \lambda_n L_n^{(\alpha)}(x) = 0, \quad \lambda_n = n. \quad (2.38)$$

We have the following relations:

$$L_n^{(\alpha)}(x) = \partial_x L_n^{(\alpha)}(x) - \partial_x L_{n+1}^{(\alpha)}(x), \quad (2.39)$$

$$x \partial_x L_n^{(\alpha)}(x) = n L_n^{(\alpha)}(x) - (n+\alpha) L_{n-1}^{(\alpha)}(x), \quad (2.40)$$

$$\partial_x L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x) = -\sum_{k=0}^{n-1} L_k^{(\alpha)}(x). \quad (2.41)$$

In particular, for  $\alpha = -k$ ,  $k = 1, 2, \dots$  (See Szegő [28, (5.2.1)]),

$$L_n^{(-k)}(x) = (-1)^k \frac{\Gamma(n-k+1)}{\Gamma(n+1)} x^k L_{n-k}^{(k)}(x), \quad n \geq k.$$

For notational convenience, we denote

$$h_n^{a,b} := \frac{\Gamma(n+1+a)}{\Gamma(n+1+a-b)}. \quad (2.42)$$

We present below some formulas related to Laguerre polynomials and fractional integrals and derivatives, which play an important role in the algorithm development and analysis later. We provide their derivations in Appendix A.

**Lemma 2.2.** For  $\mu \in \mathbb{R}^+$ , we have

$${}_0I_x^\mu \{x^\alpha L_n^{(\alpha)}(x)\} = h_n^{\alpha, -\mu} x^{\alpha+\mu} L_n^{(\alpha+\mu)}(x), \quad \alpha > -1; \quad (2.43)$$

$${}_0D_x^\mu \{x^\alpha L_n^{(\alpha)}(x)\} = h_n^{\alpha, \mu} x^{\alpha-\mu} L_n^{(\alpha-\mu)}(x), \quad \alpha > \mu - 1, \quad (2.44)$$

and

$${}_x\mathcal{I}_\infty^\mu\{e^{-x}L_n^{(\alpha)}(x)\} = e^{-x}L_n^{(\alpha-\mu)}(x), \quad \alpha > \mu - 1; \quad (2.45)$$

$${}_xD_\infty^\mu\{e^{-x}L_n^{(\alpha)}(x)\} = e^{-x}L_n^{(\alpha+\mu)}(x), \quad \alpha > -1. \quad (2.46)$$

Moreover, we have that for  $k \in \mathbb{N}$  and  $\alpha > k - 1$ ,

$$D^k\{x^\alpha e^{-x}L_n^{(\alpha)}(x)\} = \frac{\Gamma(n+k+1)}{\Gamma(n+1)}x^{\alpha-k}L_{n+k}^{(\alpha-k)}(x)e^{-x}. \quad (2.47)$$

### 3. GENERALIZED LAGUERRE FUNCTIONS

In this section, we introduce the generalized Laguerre functions (GLFs), and study its approximation properties. In what follows, the operators  ${}_0\mathcal{I}_x^{\mu,\lambda}$ ,  ${}_0D_x^{\mu,\lambda}$  on the half line should be understood as 0 in place of  $-\infty$  in (2.11) and (2.24)-(2.25).

**3.1. Definition and properties.** We first introduce the GJFs and their associated properties related to tempered fractional integrals/derivatives.

**Definition 3.1 (GLFs).** For real  $\alpha \in \mathbb{R}$  and  $\lambda > 0$ , we define the GLFs as

$$\mathcal{L}_n^{(\alpha,\lambda)}(x) := \begin{cases} x^{-\alpha}e^{-\lambda x}L_n^{(-\alpha)}(2\lambda x), & \alpha < 0, \\ e^{-\lambda x}L_n^{(\alpha)}(2\lambda x), & \alpha \geq 0 \end{cases} \quad (3.1)$$

for all  $x \in \mathbb{R}^+$  and  $n \in \mathbb{N}_0$ .

**Remark 3.1.** It's noteworthy that Zhang and Guo [30] introduce the GJFs

$$\widetilde{\mathcal{L}}_l^{(\alpha,\beta)}(x) = \begin{cases} x^{-\alpha}e^{-\frac{\beta}{2}x}L^{(-\alpha)}(\beta x), & \alpha \leq -1, \quad l \geq \bar{l}_\alpha = [-\alpha], \\ e^{-\frac{\beta}{2}x}L_l^{(\alpha)}(x), & \alpha > -1, \quad l \geq \bar{l}_\alpha = 0, \end{cases} \quad (3.2)$$

where the scaling factor  $\beta > 0$ . It is seen that we modified the definition in the range of  $0 < \alpha < 1$  (with  $\beta = 2\lambda$ ). This turns out to be essential for the numerical solution of FDEs of order  $\mu \in (0, 1)$ , as we shall see in the subsequent sections.  $\square$

We next present the basic properties of GLFs. Firstly, one verifies readily from the orthogonality (2.37) and Definition 3.1 that for  $\alpha \in \mathbb{R}$  and  $\lambda > 0$ ,

$$\int_0^\infty \mathcal{L}_n^{(\alpha,\lambda)}(x)\mathcal{L}_m^{(\alpha,\lambda)}(x)x^\alpha dx = \gamma_n^{|\alpha|,\lambda}\delta_{nm}, \quad \gamma_n^{|\alpha|,\lambda} = \frac{\gamma_n^{|\alpha|}}{(2\lambda)^{|\alpha|+1}}, \quad (3.3)$$

where  $\gamma_n^{|\alpha|}$  is defined in (2.37).

We have the following important (left) “tempered” fractional integral and derivative rules.

**Lemma 3.1.** For  $\mu, \nu, \lambda, x \in \mathbb{R}_0^+$ , we have

$${}_0\mathcal{I}_x^{\mu,\lambda}\mathcal{L}_n^{(-\nu,\lambda)}(x) = h_n^{\nu,-\mu}\mathcal{L}_n^{(-\nu-\mu,\lambda)}(x), \quad (3.4)$$

$${}_0D_x^{\mu,\lambda}\mathcal{L}_n^{(-\nu,\lambda)}(x) = h_n^{\nu,\mu}\mathcal{L}_n^{(\mu-\nu,\lambda)}(x), \quad \nu \geq \mu, \quad (3.5)$$

and

$${}_0D_x^{\mu+k,\lambda}\mathcal{L}_n^{(-\mu,\lambda)}(x) = (-2\lambda)^k h_n^{\mu,\mu}\mathcal{L}_{n-k}^{(k,\lambda)}(x), \quad n \geq k \in \mathbb{N}_0, \quad (3.6)$$

where  $h_n^{a,b}$  is defined in (2.42).

*Proof.* We obtain from (2.11) and (2.24)-(2.25) (with replacing  $-\infty$  by 0) that

$${}_0I_x^{\mu,\lambda} \mathcal{L}_n^{(-\nu,\lambda)}(x) = e^{-\lambda x} {}_0I_x^\mu \{e^{\lambda x} \mathcal{L}_n^{(-\nu,\lambda)}(x)\} = e^{-\lambda x} {}_0I_x^\mu \{x^\nu L_n^{(\nu)}(2\lambda x)\},$$

and

$${}_0D_x^{\mu,\lambda} \mathcal{L}_n^{(-\nu,\lambda)}(x) = e^{-\lambda x} {}_0D_x^\mu \{e^{\lambda x} \mathcal{L}_n^{(-\nu,\lambda)}(x)\} = e^{-\lambda x} {}_0D_x^\mu \{x^\nu L_n^{(\nu)}(2\lambda x)\}.$$

Thus, from (2.9) and Lemma 2.2, we obtain (3.4)-(3.5).

Using (3.5) and the derivative relation (2.41) (with  $\alpha = \mu$ ), we obtain

$$\begin{aligned} {}_0D_x^{\mu+k,\lambda} \mathcal{L}_n^{(-\nu,\lambda)}(x) &= {}_0D_x^{k,\lambda} {}_0D_x^{\mu,\lambda} \mathcal{L}_n^{(-\nu,\lambda)}(x) = e^{-\lambda x} D^k \{e^{\lambda x} h_n^{\mu,\mu} \mathcal{L}_n^{(0,\lambda)}(x)\} \\ &= h_n^{\mu,\mu} e^{-\lambda x} D^k \{L_n^{(0)}(2\lambda x)\} = (-2\lambda)^k h_n^{\mu,\mu} L_{n-k}^{(k)}(2\lambda x) e^{-\lambda x}. \end{aligned}$$

This leads to (3.6).  $\square$

Similarly, we have the following rules of the (right) “tempered” fractional integrals and derivatives.

**Lemma 3.2.** For  $\mu, \nu, \lambda, x \in \mathbb{R}_0^+$ , we have

$${}_xI_\infty^{\mu,\lambda} \mathcal{L}_n^{(\nu,\lambda)}(x) = (2\lambda)^{-\mu} \mathcal{L}_n^{(\nu-\mu,\lambda)}(x), \quad \nu \geq \mu, \quad (3.7)$$

$${}_xD_\infty^{\mu,\lambda} \mathcal{L}_n^{(\nu,\lambda)}(x) = (2\lambda)^\mu \mathcal{L}_n^{(\mu+\nu,\lambda)}(x). \quad (3.8)$$

*Proof.* Identities (3.7) and (3.8) can be easily derived from (2.9), (2.10) and Lemma 2.2.  $\square$

We highlight the fractional derivative formulas, which play an important role in the forthcoming algorithm and analysis.

**Theorem 3.1.** Let  $k \in \mathbb{N}$  and  $k - \nu \leq 0$ ,

$${}_{-\infty}D_x^{k,\lambda} \{\mathcal{L}_n^{(-\nu,\lambda)}(x)\} = \frac{\Gamma(n + \nu + 1)}{\Gamma(n + \nu - k + 1)} \mathcal{L}_n^{(k-\nu,\lambda)}(x), \quad (3.9)$$

$${}_xD_\infty^{k,\lambda} \{\mathcal{L}_n^{(-\nu,\lambda)}(x)\} = (-1)^k \frac{\Gamma(n + k + 1)}{\Gamma(n + 1)} \mathcal{L}_{n+k}^{(k-\nu,\lambda)}(x). \quad (3.10)$$

*Proof.* From Lemma 2.2 and relations

$${}_{-\infty}D_x^{k,\lambda} u = e^{-\lambda x} D^k \{e^{\lambda x} u\}, \quad {}_xD_\infty^{k,\lambda} u = e^{\lambda x} (-1)^k D^k \{e^{-\lambda x} u\}, \quad (3.11)$$

we obtain that for  $k - \nu \leq 0$ ,

$$\begin{aligned} {}_{-\infty}D_x^{k,\lambda} \{x^\nu \mathcal{L}_n^{(\nu,\lambda)}(x)\} &= e^{-\lambda x} D^k \{(2\lambda)^{-\nu} (2\lambda x)^\nu L_n^{(\nu)}(2\lambda x)\} \\ &\stackrel{(2.44)}{=} \frac{\Gamma(n + 1 + \nu)}{\Gamma(n + \nu - k + 1)} x^{\nu-k} L_n^{(\nu-k)}(2\lambda x) e^{-\lambda x} = \frac{\Gamma(n + 1 + \nu)}{\Gamma(n + \nu - k + 1)} \mathcal{L}_n^{(k-\nu,\lambda)}(x), \end{aligned}$$

and

$$\begin{aligned} {}_xD_\infty^{k,\lambda} \{x^\nu \mathcal{L}_n^{(\nu,\lambda)}(x)\} &= e^{\lambda x} (-1)^k D^k \{(2\lambda)^{-\nu} (2\lambda x)^\nu L_n^{(\nu)}(2\lambda x) e^{-2\lambda x}\} \\ &\stackrel{(2.47)}{=} (-1)^k \frac{\Gamma(n + k + 1)}{\Gamma(n + 1)} x^{\nu-k} L_{n+k}^{(\nu-k)}(2\lambda x) e^{-\lambda x} \\ &= (-1)^k \frac{\Gamma(n + k + 1)}{\Gamma(n + 1)} \mathcal{L}_{n+k}^{(k-\nu,\lambda)}(x). \end{aligned} \quad (3.12)$$

This ends the proof.  $\square$

Another attractive property of GLFs is that they are eigenfunctions of Sturm-Liouville problem.



**Theorem 3.2.** *Let  $s, \nu, x \in \mathbb{R}_0^+$  and  $n \in \mathbb{N}_0$ . Then,*

$$x^\nu {}_x D_\infty^{s,\lambda} \{x^{s-\nu} {}_0 D_x^{s,\lambda} \mathcal{L}_n^{(-\nu,\lambda)}(x)\} = \lambda_{n,-}^{s,\nu} \mathcal{L}_n^{(-\nu,\lambda)}(x), \quad \nu - s \geq 0, \quad (3.13)$$

and

$$x^{-\nu} {}_0 D_x^{s,\lambda} \{x^{s+\nu} {}_x D_\infty^{s,\lambda} \mathcal{L}_n^{(\nu,\lambda)}(x)\} = \lambda_{n,+}^{s,\nu} \mathcal{L}_n^{(\nu,\lambda)}(x), \quad (3.14)$$

where the corresponding eigenvalues  $\lambda_{n,-}^{s,\nu} = (2\lambda)^s h_n^{\nu,s}$  and  $\lambda_{n,+}^{s,\nu} = (2\lambda)^s h_n^{\nu+s,s}$ .

*Proof.* Due to (3.5) and (3.8),

$${}_0 D_x^{s,\lambda} \mathcal{L}_n^{(-\nu,\lambda)}(x) = h_n^{\nu,s} \mathcal{L}_n^{(s-\nu,\lambda)}(x), \quad {}_x D_\infty^{s,\lambda} \mathcal{L}_n^{(\nu-s,\lambda)}(x) = (2\lambda)^s \mathcal{L}_n^{(\nu,\lambda)}(x).$$

It's straightforward to obtain that

$$\begin{aligned} x^\nu {}_x D_\infty^{s,\lambda} \{x^{s-\nu} {}_0 D_x^{s,\lambda} \mathcal{L}_n^{(-\nu,\lambda)}(x)\} &= x^\nu {}_x D_\infty^{s,\lambda} \{x^{s-\nu} {}_0 D_x^{s,\lambda} \mathcal{L}_n^{(-\nu,\lambda)}(x)\} \\ &= h_n^{\nu,s} x^\nu {}_x D_\infty^{s,\lambda} \{x^{s-\nu} \mathcal{L}_n^{(s-\nu,\lambda)}(x)\} = h_n^{\nu,s} x^\nu {}_x D_\infty^{s,\lambda} \mathcal{L}_n^{(\nu-s,\lambda)}(x) = (2\lambda)^s h_n^{\nu,s} \mathcal{L}_n^{(-\nu,\lambda)}(x). \end{aligned}$$

Similarly, we have

$$\begin{aligned} x^{-\nu} {}_0 D_x^{s,\lambda} \{x^{s+\nu} {}_x D_\infty^{s,\lambda} \mathcal{L}_n^{(\nu,\lambda)}(x)\} &= x^{-\nu} {}_0 D_x^{s,\lambda} \{x^{s+\nu} {}_x D_\infty^{s,\lambda} \mathcal{L}_n^{(\nu,\lambda)}(x)\} \\ &= (2\lambda)^s x^{-\nu} {}_0 D_x^{s,\lambda} \{x^{s+\nu} \mathcal{L}_n^{(s+\nu,\lambda)}(x)\} = (2\lambda)^s x^{-\nu} {}_0 D_x^{s,\lambda} \mathcal{L}_n^{(-\nu-s,\lambda)}(x) = (2\lambda)^s h_n^{\nu+s,s} \mathcal{L}_n^{(\nu,\lambda)}(x). \end{aligned}$$

This ends the derivation.  $\square$

**Remark 3.2.** The above results can be viewed as an extension of the standard Sturm-Liouville problem of generalized Laguerre functions (cf. (2.38)) to the tempered fractional derivative. We derive immediately from (3.13), (3.14) and the Stirling's formula (see (3.23)) that for fixed  $s$  and  $\nu$ ,

$$\lambda_{n,-}^{s,\nu} = \lambda_{n,+}^{s,\nu} = O((2\lambda n)^s), \quad n \gg 1.$$

When  $s \rightarrow 1$  and  $\lambda = 1/2$ , it recovers the  $O(n)$  growth of eigenvalues of the standard Sturm-Liouville problem.  $\square$

### 3.2. Approximation by GLFs.

3.2.1. *Approximation by  $\{\mathcal{L}_n^{(\alpha,\lambda)}(x) : \alpha = -\nu < 0\}_{n=0}^\infty$ .* Denote by  $\mathcal{P}_N$  the set of all polynomials of degree at most  $N$ , and define the finite dimensional space

$$\mathcal{F}_N^{\nu,\lambda}(\mathbb{R}^+) := \{x^\nu e^{-\lambda x} p(x) : p \in \mathcal{P}_N\}, \quad N \in \mathbb{N}_0. \quad (3.15)$$

Define the  $L_\omega^2(\mathbb{R}^+)$  with the inner product and norm:

$$(f, g)_\omega := \int_{\mathbb{R}^+} f \bar{g} \omega dx, \quad \|f\|_\omega^2 = (f, f)_\omega, \quad (3.16)$$

where  $\omega(x)$  be a generic weight function and  $\bar{g}$  is the conjugate of the function  $g$ . In particular, we omit  $\omega$  when  $\omega \equiv 1$ .

To characterize the approximation errors, we define the non-uniformly weighted Sobolev space

$$A_{\nu,\lambda}^m(\mathbb{R}^+) := \left\{ u \in L_{\omega^{-\nu}}^2(\mathbb{R}^+) : {}_0 D_x^{\nu+k,\lambda} u \in L_{\omega^k}^2(\mathbb{R}^+), \quad k = 0, \dots, m \right\}, \quad m \in \mathbb{N}_0, \quad (3.17)$$

equipped with the norm and semi-norm

$$\|u\|_{A_{\nu,\lambda}^m} := \left( \|u\|_{\omega^{-\nu}}^2 + \sum_{k=0}^m \|{}_0 D_x^{\nu+k,\lambda} u\|_{\omega^k}^2 \right)^{1/2}, \quad |u|_{A_{\nu,\lambda}^m} := \|{}_0 D_x^{\nu+m,\lambda} u\|_{\omega^m}, \quad (3.18)$$

where the weight function  $\omega^a(x) = x^a$ .

Consider the orthogonal projection  $\pi_N^{-\nu,\lambda} : L_{\omega^{-\nu}}^2(\mathbb{R}^+) \rightarrow \mathcal{F}_N^{\nu,\lambda}(\mathbb{R}^+)$  defined by

$$(\pi_N^{-\nu,\lambda} u - u, \phi)_{\omega^{-\nu}} = 0, \quad \forall \phi \in \mathcal{F}_N^{\nu,\lambda}(\mathbb{R}^+). \quad (3.19)$$

Then, by the orthogonality (3.3),  $u$  and its  $L^2$ -orthogonal projection can be expanded as

$$u(x) = \sum_{n=0}^{\infty} \hat{u}_n \mathcal{L}_n^{(-\nu, \lambda)}(x), \quad (\pi_N^{-\nu, \lambda} u)(x) = \sum_{n=0}^N \hat{u}_n \mathcal{L}_n^{(-\nu, \lambda)}(x), \quad (3.20)$$

where

$$\hat{u}_n = (u, \mathcal{L}_n^{(-\nu, \lambda)})_{\omega^{-\nu}} / \gamma_n^{\nu, \lambda}.$$

**Theorem 3.3.** *For  $\lambda, \nu > 0$ , we have that for any  $u \in A_{\nu, \lambda}^m(\mathbb{R}^+)$  with  $m \leq N+1$ ,*

$$\|\pi_N^{-\nu, \lambda} u - u\|_{\omega^{-\nu}} \leq c (2\lambda N)^{-\frac{\nu+m}{2}} \|{}_0D_x^{\nu+m, \lambda} u\|_{\omega^m}, \quad (3.21)$$

and for any  $k \leq m$ ,

$$\|{}_0D_x^{\nu+k, \lambda} (\pi_N^{-\nu, \lambda} u - u)\|_{\omega^k} \leq c (2\lambda N)^{\frac{k-m}{2}} \|{}_0D_x^{\nu+m, \lambda} u\|_{\omega^m}, \quad (3.22)$$

where  $c \approx 1$  for large  $N$ .

*Proof.* By (3.20), we have

$$(u - \pi_N^{-\nu, \lambda} u)(x) = \sum_{n=N+1}^{\infty} \hat{u}_n \mathcal{L}_n^{(-\nu, \lambda)}(x).$$

By the orthogonality (3.3) and (3.6),

$$\|{}_0D_x^{\nu+k, \lambda} \mathcal{L}_n^{(-\nu, \lambda)}\|_{\omega^k}^2 = (-2\lambda)^{2k} (h_n^{\nu, \lambda})^2 \int_0^\infty (L_{n-k}^{(k)}(2\lambda x))^2 e^{-2\lambda x} \omega^k(x) dx = (d_{n,k}^{\nu, \lambda})^2 \gamma_{n-k}^{k, \lambda},$$

where we denoted  $d_{n,k}^{\nu, \lambda} := (2\lambda)^k h_n^{\nu, \lambda}$  and used the fact:

$$\int_0^\infty L_{n-m}^{(m)}(2\lambda x) L_{n-k}^{(k)}(2\lambda x) e^{-2\lambda x} \omega^k dx = \frac{\gamma_{n-k}^k}{(2\lambda)^{k+1}} \delta_{km} = \gamma_{n-k}^{k, \lambda} \delta_{km},$$

Thus we can obtain

$$\begin{aligned} \|\pi_N^{-\nu, \lambda} u - u\|_{\omega^{-\nu}}^2 &= \sum_{n=N+1}^{\infty} (\hat{u}_n)^2 \gamma_n^{\nu, \lambda}, \quad |\pi_N^{-\nu, \lambda} u - u|_{A_{\nu, \lambda}^k}^2 = \sum_{n=N+1}^{\infty} (\hat{u}_n d_{n,k}^{\nu, \lambda})^2 \gamma_{n-k}^{k, \lambda}, \\ |u|_{A_{\nu, \lambda}^m}^2 &= \sum_{n=m}^{\infty} (\hat{u}_n d_{n,m}^{\nu, \lambda})^2 \gamma_{n-m}^{m, \lambda}. \end{aligned}$$

Then one verifies readily that

$$\begin{aligned} \|\pi_N^{-\nu, \lambda} u - u\|_{\omega^{-\nu}}^2 &\leq \frac{\gamma_{N+1}^{\nu, \lambda}}{(d_{N+1,m}^{\nu, \lambda})^2 \gamma_{N+1-m}^{m, \lambda}} |u|_{A_{\nu, \lambda}^m}^2, \\ |\pi_N^{-\nu, \lambda} u - u|_{A_{\nu, \lambda}^k}^2 &\leq \left( \frac{d_{N+1,k}^{\nu, \lambda}}{d_{N+1,m}^{\nu, \lambda}} \right)^2 \frac{\gamma_{N+1-k}^{k, \lambda}}{\gamma_{N+1-m}^{m, \lambda}} |u|_{A_{\nu, \lambda}^m}^2. \end{aligned}$$

Recall the property of the Gamma function (see [1, (6.1.38)]):

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} \exp\left(-x + \frac{\theta}{12x}\right), \quad \forall x > 0, \quad 0 < \theta < 1. \quad (3.23)$$

One can then obtain that for any constants  $a, b$ , and for  $n \geq 1$ ,  $n+a > 1$  and  $n+b > 1$ ,

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \leq \nu_n^{a,b} n^{a-b}, \quad (3.24)$$

where

$$\nu_n^{a,b} = \exp\left(\frac{a-b}{2(n+b-1)} + \frac{1}{12(n+a-1)} + \frac{(a-b)^2}{n}\right). \quad (3.25)$$

Therefore,

$$\frac{\gamma_{N+1}^{\nu,\lambda}}{(d_{N+1,m}^{\nu,\lambda})^2 \gamma_{N+1-m}^{m,\lambda}} = \frac{\Gamma(N+2-m)}{(2\lambda)^{\nu+m} \Gamma(N+2+\nu)} \leq (2\lambda)^{-\nu-m} \nu_n^{2-m,2+\nu} N^{-\nu-m},$$

$$\left( \frac{d_{N+1,k}^{\nu,\lambda}}{d_{N+1,m}^{\nu,\lambda}} \right)^2 \frac{\gamma_{N+1-k}^{k,\lambda}}{\gamma_{N+1-m}^{m,\lambda}} = \frac{(2\lambda)^k \Gamma(N+2-m)}{(2\lambda)^m \Gamma(N+2-k)} < (2\lambda)^{k-m} \nu_n^{2-m,2-k} N^{k-m},$$

where  $\nu_n^{2-m,2+\nu} \approx 1$  and  $\nu_n^{2-m,2-k} \approx 1$  for fixed  $m$  and  $n \geq N \gg 1$ . Then (3.21)-(3.22) follow.  $\square$

3.2.2. *Approximation by  $\{\mathcal{L}_n^{(\alpha,\lambda)}(x) : \alpha = \nu \geq 0\}_{n=0}^\infty$ .* Introduce the non-uniformly weighted Sobolev space:

$$B_{\nu,\lambda}^r(\mathbb{R}^+) := \left\{ u \in L_{\omega^\nu}^2(\mathbb{R}^+) : {}_x D_\infty^{s,\lambda} u \in L_{\omega^{\nu+s}}^2(\mathbb{R}^+), 0 \leq s \leq r \right\}, \quad r \in \mathbb{R}_0^+, \quad (3.26)$$

endowed with norm and semi-norm

$$\|u\|_{B_{\nu,\lambda}^r} := \left( \|u\|_{\omega^\nu}^2 + |u|_{B_{\nu,\lambda}^r}^2 \right)^{1/2}, \quad |u|_{B_{\nu,\lambda}^r} := \|{}_x D_\infty^{\nu+r,\lambda} u\|_{\omega^{\nu+r}}. \quad (3.27)$$

Consider the orthogonal projection  $\Pi_N^{\nu,\lambda} : L_{\omega^\nu}^2(\mathbb{R}^+) \rightarrow \mathcal{F}_N^{0,\lambda}(\mathbb{R}^+)$ , defined by

$$(\Pi_N^{\nu,\lambda} u - u, \phi)_{\omega^\nu} = 0, \quad \forall \phi \in \mathcal{F}_N^{0,\lambda}(\mathbb{R}^+), \quad \nu > -1, \quad (3.28)$$

**Theorem 3.4.** *Let  $\lambda, r, \nu > 0$ . For any  $u \in B_{\nu,\lambda}^r(\mathbb{R}^+)$  with  $0 \leq s \leq r \leq N$ , we have*

$$\|{}_x D_\infty^{s,\lambda} \{\Pi_N^{\nu,\lambda} u - u\}\|_{\omega^{\nu+s}} \leq c (2\lambda N)^{\frac{s-r}{2}} \|{}_x D_\infty^{r,\lambda} u\|_{\omega^{\nu+r}}, \quad (3.29)$$

where  $c \approx 1$  for large  $N$ .

*Proof.* Note that by definition,

$$u - \Pi_N^{\nu,\lambda} u = \sum_{n=N+1}^\infty \hat{u}_n \mathcal{L}_n^{(\nu,\lambda)}(x), \quad \hat{u}_n = (u, \mathcal{L}_n^{(\nu,\lambda)})_{\omega^\nu} / \gamma_n^{\nu,\lambda}.$$

Then by (3.8), and the orthogonality,

$$\|{}_x D_\infty^{s,\lambda} \mathcal{L}_n^{(\nu,\lambda)}\|_{\omega^{\nu+s}}^2 = \|\mathcal{L}_n^{(\nu+s,\lambda)}\|_{\omega^{\nu+s}}^2 = \gamma_n^{\nu+s,\lambda},$$

we can derive

$$\begin{aligned} \|\Pi_N^{\nu,\lambda} u - u\|_{B_{\nu,\lambda}^s}^2 &= \left\| \sum_{n=N+1}^\infty \hat{u}_n (2\lambda)^s \mathcal{L}_n^{(\nu+s,\lambda)} \right\|_{\omega^{\nu+s}}^2 = \sum_{n=N+1}^\infty (\hat{u}_n)^2 (2\lambda)^{2s} \gamma_n^{\nu+s,\lambda}, \\ |u|_{B_{\nu+r,\lambda}^r}^2 &= \left\| \sum_{n=0}^\infty \hat{u}_n (2\lambda)^r \mathcal{L}_n^{(\nu+r,\lambda)} \right\|_{\omega^{\nu+r}}^2 = \sum_{n=0}^\infty (\hat{u}_n)^2 (2\lambda)^{2r} \gamma_n^{\nu+r,\lambda}. \end{aligned}$$

Then,

$$\|\Pi_N^{\nu,\lambda} u - u\|_{B_{\nu,\lambda}^s}^2 = \sum_{n=N+1}^\infty (\hat{u}_n)^2 \gamma_n^{\nu+s,\lambda} \leq (2\lambda)^{2s-2r} \frac{\gamma_{N+1}^{\nu+s,\lambda}}{\gamma_{N+1}^{\nu+r,\lambda}} \sum_{n=N+1}^\infty (\hat{u}_n)^2 \gamma_n^{\nu+r,\lambda},$$

where by (3.24)-(3.25), we obtain

$$\frac{\gamma_{N+1}^{\nu+s,\lambda}}{\gamma_{N+1}^{\nu+r,\lambda}} = \frac{(2\lambda)^r \Gamma(N+\nu+s+2)}{(2\lambda)^s \Gamma(N+\nu+r+2)} \leq c (2\lambda)^{r-s} N^{s-r}.$$

Consequently, we have

$$\|{}_x D_\infty^{s,\lambda} \{\Pi_N^{\nu,\lambda} u - u\}\|_{\omega^{\nu+s}} \leq c (2\lambda N)^{\frac{s-r}{2}} |u|_{B_{\nu,\lambda}^r}.$$

This ends the proof.  $\square$

**3.3. A model problem and numerical results.** In what follows, we consider the GLF approximation to a model tempered fractional equation of order  $s \in [k-1, k)$  with  $k \in \mathbb{N}$ :

$${}_0D_x^{s,\lambda}u(x) = f(x), \quad x \in \mathbb{R}^+, \quad \lambda > 0; \quad u^{(j)}(0) = 0, \quad j = 0, 1, \dots, k-1, \quad (3.30)$$

where  $f \in L^2(\mathbb{R}^+)$  is a given function. Using the fractional derivative relation (2.7), one can find

$$u(x) = {}_0I_x^{s,\lambda}f(x) + \sum_{i=1}^k c_i x^{s-i} e^{-\lambda x},$$

where  $\{c_i\}$  can be determined by the conditions at  $x = 0$ . In fact, we have all  $c_i = 0$ , and

$$u(x) = {}_0I_x^{s,\lambda}f(x) = \frac{e^{-\lambda x}}{\Gamma(s)} \int_0^x (x-\tau)^{s-1} e^{\lambda \tau} f(\tau) d\tau = \frac{x^s}{\Gamma(s)} \int_0^1 (1-t)^{s-1} e^{-\lambda(1-t)x} f(xt) dt. \quad (3.31)$$

We see that if  $f(x)$  is smooth, then  $u(x) = x^s F(x)$ , where  $F(x)$  is smoother than  $f(x)$ . With this understanding, we construct the GLF Petrov-Galerkin approximation as: find  $u_N \in \mathcal{F}_N^{s,\lambda}(\mathbb{R}^+)$  (defined in (3.15)) such that

$$({}_0D_x^{s,\lambda}u_N, v_N) = (f, v_N), \quad \forall v_N \in \mathcal{F}_N^{0,\lambda}(\mathbb{R}^+). \quad (3.32)$$

We expand  $f$  and  $u_N$  as

$$f(x) = \sum_{n=0}^{\infty} \hat{f}_n \mathcal{L}_n^{(0,\lambda)}(x), \quad u_N = \sum_{n=0}^N \hat{u}_n \mathcal{L}_n^{(s,\lambda)}(x). \quad (3.33)$$

Using the derivative relation (3.5), we find immediately that  $\hat{u}_n = \frac{\gamma_n^{0,\lambda}}{h_n^{s,s}} \hat{f}_n$  for  $n = 0, 1, \dots, N$ , which also implies  ${}_0D_x^{s,\lambda}u_N = \pi_N^{0,\lambda}f$ .

Moreover, we can show that the numerical solution  $u_N$  is precisely the orthogonal projection in the following sense:

$$(u_N - u, w_N)_{\omega^{-s}} = 0, \quad \forall w_N \in \mathcal{F}_N^{s,\lambda}(\mathbb{R}^+). \quad (3.34)$$

To this end, we first show

$$(u_N - u, {}_x D_{\infty}^{s,\lambda}v_N) = ({}_0D_x^{s,\lambda}u_N - {}_0D_x^{s,\lambda}u, v_N) = 0, \quad \forall v_N \in \mathcal{F}_N^{0,\lambda}(\mathbb{R}^+). \quad (3.35)$$

Indeed, thanks to  $u^{(j)}(0) = 0$  for  $j = 0, \dots, k-1$ , we have

$${}_0D_x^{s,\lambda}\{u_N - u\} = e^{-\lambda x} {}_0D_x^s\{e^{\lambda x}(u_N - u)\} = e^{-\lambda x} {}_0I_x^{k-s} {}_0D_x^k\{e^{\lambda x}(u_N - u)\}.$$

Then,

$$\begin{aligned} ({}_0D_x^{s,\lambda}u_N - {}_0D_x^{s,\lambda}u, v_N) &= ({}_0I_x^{k-s} {}_0D_x^k\{e^{\lambda x}(u_N - u)\}, e^{-\lambda x}v_N) \\ &= (e^{\lambda x}(u_N - u), (-1)^k {}_x D_{\infty}^k I_{\infty}^{k-s,\lambda}v_N) = (u_N - u, {}_x D_{\infty}^{s,\lambda}v_N), \end{aligned}$$

so (3.35) is valid. In addition, thanks to Lemma 2.2 and (2.10), we have

$${}_x D_{\infty}^{s,\lambda} \mathcal{L}_n^{(0,\lambda)}(x) = e^{\lambda x} {}_x D_{\infty}^s\{e^{-2\lambda x} L_n^{(0)}(2\lambda x)\} = (2\lambda)^s e^{-\lambda x} L_n^{(s)}(2\lambda x) = (2\lambda)^s x^{-s} \mathcal{L}_n^{(-s,\lambda)}(x).$$

Hence, (3.34) is valid.

Thanks to (3.34), we derive from Theorem 3.3 the following estimate where the convergence rate only depends on the regularity of the source term.

**Theorem 3.5.** *Let  $u$  and  $u_N$  be respectively the solutions of (3.30) and (3.32). Then for  ${}_0D_x^{m,\lambda}f \in L_{\omega,m}^2(I)$  with  $m \in \mathbb{N}_0$ , we have*

$$\|u - u_N\|_{\omega^{-s}} \leq c (2\lambda N)^{-\frac{s+m}{2}} \|{}_0D_x^{s+m,\lambda}u\|_{\omega^m} = c (2\lambda N)^{-\frac{s+m}{2}} \|{}_0D_x^{m,\lambda}f\|_{\omega^m}, \quad (3.36)$$

where  $c \approx 1$  for large  $N$ .

We provide some numerical results to illustrate the convergence behaviour. We take  $f(x) = e^{-x} \sin x$  and then evaluate the exact solution by (3.31). Note that as  ${}_0D_x^{m,\lambda} f = e^{-\lambda x} {}_0D_x^m \{e^{\lambda x} f\}$ , a direct calculation leads to

$$e^{-\lambda x} {}_0D_x^m \{e^{\lambda x} f\} = e^{-\lambda x} \sum_{k=0}^m \binom{m}{k} \lambda^{m-k} e^{\lambda x} {}_0D_x^k f = \sum_{k=0}^m \binom{m}{k} \lambda^{m-k} {}_0D_x^k f.$$

We infer from (3.36) that the spectral accuracy can be achieved by the GLF approximation. Indeed, we observe from Figure 3.1 such a convergence behaviour.

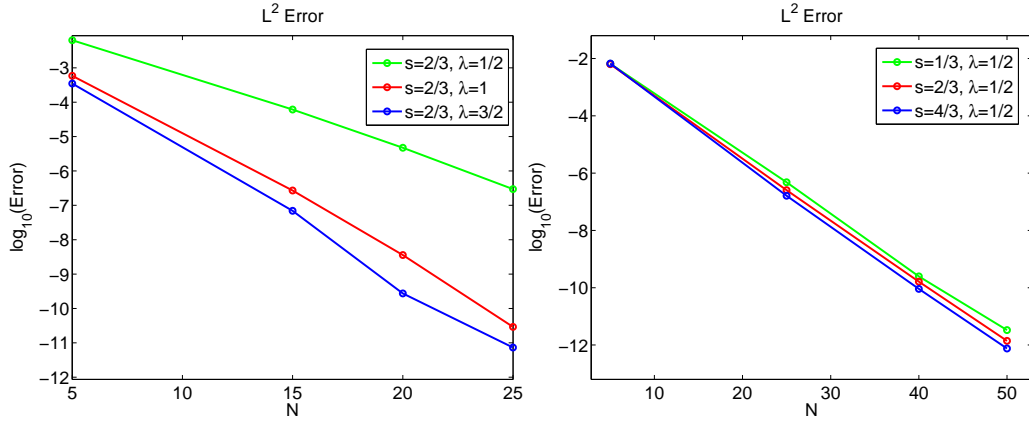


FIGURE 3.1. Convergence of the GLF approximation to (3.30) with  $f(x) = e^{-x} \sin x$ .

#### 4. APPLICATION TO TEMPERED FRACTIONAL DIFFUSION EQUATION ON THE HALF LINE

In this section, we apply the GLFs to approximate a tempered fractional diffusion equation on the half-line.

**4.1. The tempered fractional diffusion equation on the half line.** Consider the tempered fractional diffusion equation of order  $\mu \in (0, 1)$  on the half line:

$$\begin{cases} \partial_t u(x, t) + {}_0D_x^{\mu, \lambda} u(x, t) - \lambda^\mu u(x, t) = f(x, t), & (x, t) \in \mathbb{R}^+ \times (0, T], \\ u(0, t) = 0, \quad \lim_{x \rightarrow \infty} u(x, t) = 0, & 0 < t \leq T, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^+. \end{cases} \quad (4.1)$$

This equation models the particles jumping on the half line  $\mathbb{R}^+$  with the probability density function (see [23, (8)]):

$$f_\varepsilon(x) = C_\varepsilon^{-1} x^{-\mu-1} e^{-\lambda x} \mathbf{1}_{(\varepsilon, \infty)}(x), \quad 0 < \mu < 1.$$

**Remark 4.1.** We note that equation (4.1) is equivalent to the half line form of the TFDE (1.2) in which

$$\partial_{+,x}^{\mu, \lambda} u = {}_0D_x^{\mu, \lambda} u - \lambda^\mu u, \quad 0 < \mu < 1.$$

Indeed, we can show that for  $\mu \in (0, 1)$  and real  $\lambda > 0$ ,

$${}_0D_x^{\mu, \lambda} u = e^{-\lambda x} {}_0D_x^\mu \{e^{\lambda x} u(x)\} = e^{-\lambda x} {}_{-\infty}D_x^\mu \{e^{\lambda x} \tilde{u}(x)\}, \quad x \in \mathbb{R}^+,$$

where  $\tilde{u} = u$  for  $x \in \mathbb{R}^+$  and  $\tilde{u} = 0$  for  $x \in (-\infty, 0)$ . Moreover, we have

$$\begin{aligned} \mathcal{F}[e^{-\lambda x} {}_{-\infty}D_x^\mu \{e^{\lambda x} \tilde{u}(x)\}](\omega) &= \int_{\mathbb{R}} D_{-\infty} I_x^{1-\mu} \{e^{\lambda x} \tilde{u}(x)\} e^{-(\lambda+i\omega)x} dx \\ &= (\lambda + i\omega) \int_{\mathbb{R}} {}_{-\infty}I_x^{1-\mu} \{e^{\lambda x} \tilde{u}(x)\} e^{-(\lambda+i\omega)x} dx \\ &= (\lambda + i\omega) \int_{\mathbb{R}} e^{\lambda x} \tilde{u}(x) {}_xI_\infty^{1-\mu} e^{-(\lambda+i\omega)x} dx \\ &= (\lambda + i\omega)^\mu \mathcal{F}[\tilde{u}](\omega) \stackrel{(2.20)}{=} \mathcal{F}[{}_{-\infty}D_x^{\mu,\lambda} \tilde{u}(x)](\omega). \end{aligned}$$

This implies  $\tilde{u} \in W_{\lambda}^{\mu,2}(\mathbb{R})$  and the extended tempered fractional derivative  ${}_0D_x^{\mu,\lambda}u$  can be understood in the sense of the original definition in [23].  $\square$

**4.2. Spectral-Galerkin scheme.** Define

$$H^{\mu,\lambda}(\mathbb{R}^+) := \{v \in L^2(\mathbb{R}^+) : {}_0D_x^{s,\lambda}v \in L^2(\mathbb{R}^+), 0 < s \leq \mu\}, \quad s, \lambda > 0, \quad \mu \in (0, 1),$$

with the semi norm and norm

$$|v|_{\mu,\lambda} = \|{}_0D_x^{\mu,\lambda}v\|, \quad \|v\|_{\mu,\lambda} = (\|v\|^2 + |v|_{\mu,\lambda}^2)^{1/2}.$$

Furthermore, let  $H_0^{\mu,\lambda}(\mathbb{R}^+)$  be the closure of  $C_0^\infty(\mathbb{R}^+)$  with respect to the norm  $\|\cdot\|_{\mu,\lambda}$ .

Thanks to the homogeneous boundary condition and (2.30), a weak form of (4.1) is to find  $u(\cdot, t) \in H_0^{\mu,\lambda}(\mathbb{R}^+)$  such that

$$(\partial_t u(\cdot, t), v) + a_\mu(u(\cdot, t), v) = (f(\cdot, t), v), \quad \forall v \in H_0^{\mu,\lambda}(\mathbb{R}^+), \quad (4.2)$$

with  $u(x, 0) = u_0(x)$ , where

$$a_\mu(u, v) := ({}_0D_x^{\mu,\lambda}u, v) - \lambda^\mu(u, v). \quad (4.3)$$

The semi-discrete Galerkin approximation scheme is to find  $u_N(\cdot, t) \in \mathcal{F}_N^{\nu,\lambda}(\mathbb{R}^+)$  such that

$$(\partial_t u_N(\cdot, t), v) + a_\mu(u_N(\cdot, t), v) = (f(\cdot, t), v), \quad \forall v \in \mathcal{F}_N^{\nu,\lambda}(\mathbb{R}^+), \quad (4.4)$$

with

$$u_N(x, 0) = u_{0,N}(x) = \sum_{n=0}^N c_{0,n} \mathcal{L}_n^{(-\nu,\lambda)}(x).$$

Here, we choose  $\max\{0, \mu - \frac{1}{2}\} < \nu \leq 1$  so that  $u(0, t) = 0$ .

Now, we set

$$u_N(x, t) = \sum_{n=0}^N c_n(t) \varphi_n(x), \quad \varphi_n(x) := \mathcal{L}_n^{(-\nu,\lambda)}(x). \quad (4.5)$$

We derive from the scheme (4.4) that

$$\mathbf{M} \frac{d}{dt} \vec{c}(t) + \mathbf{A} \vec{c}(t) = \vec{f}(t); \quad \vec{c}(0) = \vec{c}_0. \quad (4.6)$$

where for fixed  $t > 0$ , vectors

$$\begin{aligned} \vec{c}(t) &= (c_0(t), c_1(t), \dots, c_N(t))^T, \quad \vec{c}_0 = (c_{0,0}(t), c_{0,1}(t), \dots, c_{0,N}(t))^T, \\ \vec{f}(t) &= (f_0(t), f_1(t), \dots, f_N(t))^T, \quad f_n(t) = (f, \varphi_n), \quad 0 \leq n \leq N. \end{aligned} \quad (4.7)$$

and

$$\mathbf{M}_{mn} = (\varphi_n, \varphi_m), \quad \mathbf{A}_{mn} = a_\mu(\varphi_n, \varphi_m), \quad m, n = 0, 1, 2, \dots, N. \quad (4.8)$$

**4.3. Numerical results.** For clarity, we test three cases:

(i).  $u(x, t) = xe^{-\lambda x} \cos(t)$ . By a direct calculation, the source term is given by

$$f(x, t) = -xe^{-\lambda x} \sin(t) + \left( \frac{\Gamma(2)}{\Gamma(2-\mu)} x^{1-\mu} - \lambda^\mu x \right) e^{-\lambda x} \cos(t).$$

The left of Figure 4.1 illustrates that error decays to zero dramatically when using the spectral method with basis  $\mathcal{L}_n^{(-1, \lambda)}(x)$  in the space and the third-order explicit Runge-Kutta method in time direction for  $\lambda = 2/3$ ,  $\mu = 2/3$ .

(ii).  $f(x, t) = \cos(x)e^{-x} \sin(t)$  and fix  $\lambda$  and  $\mu$  as before, the right graph verifies that the solution is singular even though  $f(x, t)$  is a smooth function.

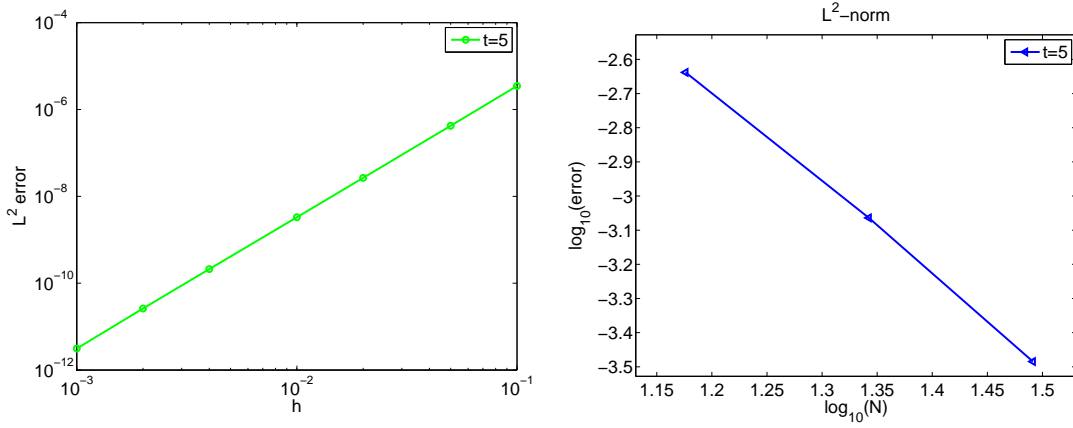


FIGURE 4.1. Left:  $u = x \exp(-\lambda x) \cos(t)$ . Right:  $f = \cos(x) \exp(-x) \sin(t)$ .

(iii). Consider the case  $f(x, t) \equiv 0$ . Let  $\mu = 2/3$ ,  $\lambda = 2/3$  in (4.1). The left of the Figure 4.2 exhibits the evolution of the tempered fractional diffusion model with the initial distribution  $u_0(x) = xe^{-x}$ . The right describes the approximate rate by diverse basis  $\mathcal{L}_n^{(-\nu, \lambda)}(x)$  at time  $t = 10$ .

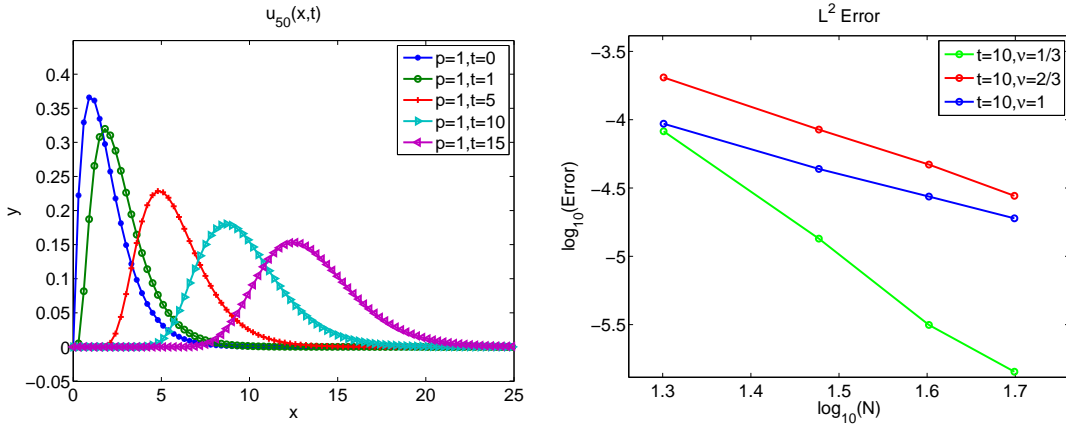


FIGURE 4.2. TFDEs:  $f \equiv 0$ ,  $\lambda = 2/3$ ,  $\mu = 2/3$ .

## 5. TEMPERED FRACTIONAL DIFFUSION EQUATION ON THE WHOLE LINE

In this section, we present a spectral-element method with two-subdomains for the tempered fractional diffusion equation on the whole line originally proposed by [23].

**5.1. Tempered fractional diffusion equation.** We consider the tempered fractional diffusion equation of order  $\mu \in (k-1, k)$ ,  $k = 1, 2$  on the whole line:

$$\begin{cases} \partial_t u(x, t) = (-1)^k C_T \{p \partial_{+,x}^{\mu,\lambda} + q \partial_{-,x}^{\mu,\lambda}\} u(x, t) + f(x, t), \\ u(x, 0) = u_0(x), \quad \lim_{|x| \rightarrow \infty} u(x, t) = 0, \end{cases} \quad (5.1)$$

where  $p, q$  are constants such that  $0 \leq p, q \leq 1$ ,  $p + q = 1$ ,  $C_T$  is a constant and the fractional operators are

- For  $0 < \mu < 1$ ,

$$\partial_{+,x}^{\mu,\lambda} u = -\infty D_x^{\mu,\lambda} u - \lambda^\mu u, \quad \partial_{-,x}^{\mu,\lambda} u = {}_x D_\infty^{\mu,\lambda} u - \lambda^\mu u; \quad (5.2)$$

- For  $1 < \mu < 2$ ,

$$\partial_{+,x}^{\mu,\lambda} u = -\infty D_x^{\mu,\lambda} u - \mu \lambda^{\mu-1} \partial_x u - \lambda^\mu u, \quad \partial_{-,x}^{\mu,\lambda} u = {}_x D_\infty^{\mu,\lambda} u + \mu \lambda^{\mu-1} \partial_x u - \lambda^\mu u. \quad (5.3)$$

**5.2. A two-domain spectral-element method.** Let  $\Lambda := (a, b)$ ,  $-\infty \leq a < b \leq \infty$ , and  $\omega > 0$  be a generic weight function. For any  $m \in \mathbb{N}$  and a given weight function  $\omega$ , we denote

$$H_\omega^m(\Lambda) := \{v \in L_\omega^2(\Lambda) : \partial_x^k v \in L_\omega^2(\Lambda), \quad 0 < k \leq m\}$$

with the semi-norm and norm

$$|v|_{m,\omega,\Lambda} = \|\partial_x^m v\|_{\omega,\Lambda}, \quad \|v\|_{m,\omega,\Lambda} = \left( \sum_{k=0}^m |v|_{k,\omega,\Lambda}^2 \right)^{1/2}.$$

In particular, we omit the subscript  $\omega$  when  $\omega \equiv 1$ .

Moreover, for real  $r > 0$ , we define

$$H^{r,\lambda}(\mathbb{R}) := \{v \in L^2(\mathbb{R}) : -\infty D_x^{r,\lambda} v \in L^2(\mathbb{R})\}$$

with the semi norm and norm

$$|v|_{r,\lambda} = \|-\infty D_x^{r,\lambda} v\|, \quad \|v\|_{r,\lambda} = (\|v\|^2 + |v|_{r,\lambda}^2)^{1/2}.$$

We decompose the whole line as follows

$$\mathbb{R} = \Lambda_1 \cup \Lambda_2, \quad \Lambda_1 = (-\infty, 0), \quad \Lambda_2 = [0, \infty),$$

and denote  $u_{\Lambda_j}(x, t) := u(x, t)|_{\Lambda_j}$ ,  $j = 1, 2$ . Introduce the approximation space:

$$V_{\mathbf{N}}^\lambda(\mathbb{R}) := \left\{ \phi \in C(\mathbb{R}) : \phi_{\Lambda_i}(x) = e^{-\lambda|x|} p, \quad p_{\Lambda_i} \in \mathcal{P}_{N_i}(\Lambda_i), \quad i = 1, 2 \right\}, \quad \mathbf{N} = (N_1, N_2), \quad (5.4)$$

and define

$$\phi^*(x) = e^{-\lambda|x|}, \quad \phi_{n_1}^-(x) = \begin{cases} \mathcal{L}_{n_1}^{(-1,\lambda)}(-x), & x \leq 0, \\ 0, & x > 0, \end{cases} \quad \phi_{n_2}^+(x) = \begin{cases} 0, & x \leq 0, \\ \mathcal{L}_{n_2}^{(-1,\lambda)}(x), & x > 0, \end{cases} \quad (5.5)$$

where  $\mathcal{L}_n^{(-1,\lambda)}(x) = e^{-\lambda x} x L_n^{(1)}(2\lambda x)$ . One verifies readily that

$$V_{\mathbf{N}}^\lambda(\mathbb{R}) = \text{span}\{\phi^*(x); \phi_{n_1}^-(x), \quad 0 \leq n_1 \leq N_1 - 1; \phi_{n_2}^+(x), \quad 0 \leq n_2 \leq N_2 - 1\}. \quad (5.6)$$

Then, our semi-discrete spectral-Galerkin method is to find  $u_{\mathbf{N}}(\cdot, t) \in V_{\mathbf{N}}^\lambda(\mathbb{R})$  such that

$$\begin{cases} (\partial_t u_{\mathbf{N}}(\cdot, t), v) + a_{pq}^\mu (u_{\mathbf{N}}(\cdot, t), v) = (f(\cdot, t), v), & \forall v \in V_{\mathbf{N}}^\lambda(\mathbb{R}), \\ (u_{\mathbf{N}}(\cdot, 0), v) = (u_0, v), & \forall v \in V_{\mathbf{N}}^\lambda(\mathbb{R}), \end{cases} \quad (5.7)$$



where the bilinear form  $a_{pq}^\mu(\cdot, \cdot)$  is defined by

$$a_{pq}^\mu(u, v) := \begin{cases} p(-_\infty D_x^{\mu, \lambda} u, v) + q(u, -_\infty D_x^{\mu, \lambda} v) - \lambda^\mu(u, v), & 0 < \mu < 1, \\ -\{p(-_\infty D_x^{\mu-1, \lambda} u, {}_x D_\infty^{1, \lambda} v) + q({}_x D_\infty^{1, \lambda} u, -_\infty D_x^{\mu-1, \lambda} v)\} \\ \quad + \lambda^\mu(u, v) + (p - q)\mu\lambda^{\mu-1}(\partial_x u, v), & 1 < \mu < 2. \end{cases} \quad (5.8)$$

We provide below some details of the algorithm.

$$\begin{aligned} u_N(x, t) &= c^*(t)\phi^*(x) + \sum_{n_1=0}^{N_1-1} c_{n_1}^-(t)\phi_{n_1}^-(x) + \sum_{n_2=0}^{N_2-1} c_{n_2}^+(t)\phi_{n_2}^+(x), \\ u_N(x, 0) &= c_0^*\phi^*(x) + \sum_{n_1=0}^{N_1-1} c_{0, n_1}^-\phi_{n_1}^-(x) + \sum_{n_2=0}^{N_2-1} c_{0, n_2}^+\phi_{n_2}^+(x). \end{aligned} \quad (5.9)$$

Let  $H(x)$  be the Heaviside function as before. Thanks to the tempered fractional derivative and integral relations with GLFs, and a reflected mapping from positive half line  $\mathbb{R}^+$  to negative half line  $\mathbb{R}^-$ , we can derive the following identities (see Appendix B):

$$\begin{aligned} {}_x D_\infty^{1, \lambda} \phi^*(x) &= -2\lambda e^{-\lambda x} H(x), \quad {}_x D_\infty^{1, \lambda} \phi_{n_1}^-(x) = (n_1 + 1) \mathcal{L}_{n_1+1}^{(0, \lambda)}(-x) H(-x), \\ -_\infty D_x^{s, \lambda} \phi^*(x) &= \begin{cases} (2\lambda)^s e^{\lambda x}, & x \leq 0, \\ \frac{2\lambda e^{\lambda x}}{\Gamma(1-s)} \int_x^\infty \frac{e^{-2\lambda t}}{t^s} dt, & x > 0, \end{cases} \\ -_\infty D_x^{s, \lambda} \phi_{n_1}^-(x) &= \begin{cases} -(2\lambda)^{s-1} (n_1 + 1) L_{n_1+1}^{(s-1)}(-2\lambda x) e^{\lambda x}, & x \leq 0, \\ -e^{\lambda x} \frac{n_1 + 1}{\Gamma(1-s)} \int_x^\infty \frac{L_{n_1+1}^{(0)}(2\lambda(t-x)) e^{-2\lambda t}}{t^s} dt, & x > 0, \end{cases} \quad (5.10) \\ {}_x D_\infty^{1, \lambda} \phi_{n_2}^+(x) &= -(n_2 + 1) \mathcal{L}_{n_2+1}^{(0, \lambda)}(x) H(x), \\ -_\infty D_x^{s, \lambda} \phi_{n_2}^+(x) &= \frac{\Gamma(n_2 + 2)}{\Gamma(n_2 + 2 - s)} x^{1-s} \mathcal{L}_{n_2}^{(1-s, \lambda)}(x) H(x). \end{aligned}$$

Then (5.7) leads to the system

$$\mathbf{M} \frac{d}{dt} \vec{\mathbf{C}}(t) + \mathbf{A} \vec{\mathbf{C}}(t) = \vec{\mathbf{F}}(t), \quad (5.11)$$

where

$$\begin{aligned} \vec{\mathbf{C}}(t) &= (c^*(t), \vec{\mathbf{C}}^-(t), \vec{\mathbf{C}}^+(t))^T, \quad \vec{\mathbf{F}}(t) = (f^*(t), \vec{\mathbf{F}}^-(t), \vec{\mathbf{F}}^+(t))^T, \\ \vec{\mathbf{C}}^-(t) &= (c_0^-(t), c_1^-(t), \dots, c_{N_1-1}^-(t))^T, \quad \vec{\mathbf{C}}^+(t) = (c_0^+(t), c_1^+(t), \dots, c_{N_2-1}^+(t))^T, \\ \vec{\mathbf{F}}^-(t) &= (f_0^-(t), f_1^-(t), \dots, f_{N_1-1}^-(t))^T, \quad \vec{\mathbf{F}}^+(t) = (f_0^+(t), f_1^+(t), \dots, f_{N_2-1}^+(t))^T, \\ f^*(t) &= (f, \phi^*), \quad f_{n_1}^-(t) = (f, \phi_{n_1}^-), \quad f_{n_2}^+(t) = (f, \phi_{n_2}^+), \quad 0 \leq n_i \leq N_i - 1, \quad i = 1, 2, \end{aligned}$$

and the matrices

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{1 \times 1}^{(*, *)} & \mathbf{M}_{1 \times N_1}^{(*, -)} & \mathbf{M}_{1 \times N_2}^{(*, +)} \\ \mathbf{M}_{N_1 \times 1}^{(-, *)} & \mathbf{M}_{N_1 \times N_1}^{(-, -)} & \mathbf{M}_{N_1 \times N_2}^{(-, +)} \\ \mathbf{M}_{N_2 \times 1}^{(+, *)} & \mathbf{M}_{N_2 \times N_1}^{(+, -)} & \mathbf{M}_{N_2 \times N_2}^{(+, +)} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{1 \times 1}^{(*, *)} & \mathbf{A}_{1 \times N_1}^{(*, -)} & \mathbf{A}_{1 \times N_2}^{(*, +)} \\ \mathbf{A}_{N_1 \times 1}^{(-, *)} & \mathbf{A}_{N_1 \times N_1}^{(-, -)} & \mathbf{A}_{N_1 \times N_2}^{(-, +)} \\ \mathbf{A}_{N_2 \times 1}^{(+, *)} & \mathbf{A}_{N_2 \times N_1}^{(+, -)} & \mathbf{A}_{N_2 \times N_2}^{(+, +)} \end{pmatrix}, \quad (5.12)$$

with the entries

$$\begin{aligned} \mathbf{M}_{c \times d}^{(a,b)}(i+1, j+1) &= (\phi_j^b, \phi_i^a), \quad \mathbf{A}_{c \times d}^{(a,b)}(i+1, j+1) = a_{pq}^\mu (\phi_j^b, \phi_i^a), \\ a, b &= *, -, +, \quad c, d = 1, N_1, N_2, \quad 0 \leq i \leq c-1, \quad 0 \leq j \leq d-1, \end{aligned}$$

and  $\vec{\mathbf{C}}(0)$  is determined by the initial data.

The proof of the tempered derivative relation (5.10), and the detail on the entries of the matrix  $\mathbf{A}$  can be found in Appendix B. Base on the semi-discrete scheme (5.11), we further use the third-order explicit Runge-Kutta method in time direction with step size  $h = 10^{-3}$  to numerically solve the problem.

**5.3. Numerical results.** We solve (5.1) with  $C_T = 1$  and  $u_0 = 10e^{-5|x|}$  as the initial distribution by using the method presented in the previous section. We first test the accuracy of our method. In Figure 5.1, we plot the convergence rate of the spectral method at  $T = 5$  with fixed time step  $h = 10^{-3}$ , in which  $f(x, t) \equiv 0$  and  $f(x, t) = \cos t e^{-x^2}$  are the resource terms of the left and the right respectively.

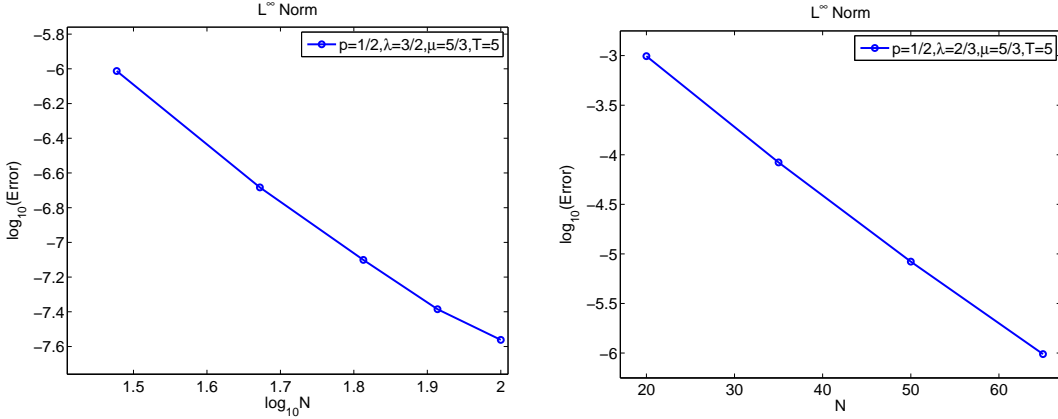


FIGURE 5.1. Left:  $f(x, t) \equiv 0$ . Right:  $f(x, t) = \cos t e^{-x^2}$ .

Next, we examine behaviors of the solution under various situations. In Figure 5.2, we plot the snapshots at different times of the tempered fractional diffusion with  $p = 1/3$ ,  $q = 2/3$  and  $p = 3/4$ ,  $q = 1/4$ , respectively. The case with  $p = q = 1/2$  is plotted in Figure 5.3.

- The parameters  $p$  and  $q$  reflect the directional preference of the particle jumping. More precisely, if  $p > q$ , the particles tend to jump to the right, and if  $p < q$ , the particles tend to jump to the left, see Figure 5.2. In particular,  $p = q$  produces a symmetric profile in the case of  $f(x, t) = 0$ , see the left in Figure 5.3.
- The parameter  $\lambda$  determines the probability of the jump distance of the particles. A larger  $\lambda$  indicates a shorter jump distance, see the right of Figure 5.3.
- To compare with the usual fractional diffusion equation, i.e.,  $\lambda = 0$ , we plot in Figure 5.4 the particle distributions of the usual fractional diffusion and the tempered fractional diffusion with initial distribution  $u_0(x) = 10e^{-4x^2}$  at time  $t = 10$ . We observe that the tail of the tempered fractional diffusion behaves like  $|x|^{-\mu-1}e^{-\lambda|x|}$  for large  $|x|$  while that of the usual fractional diffusion behaves like  $|x|^{-\mu-1}$ .

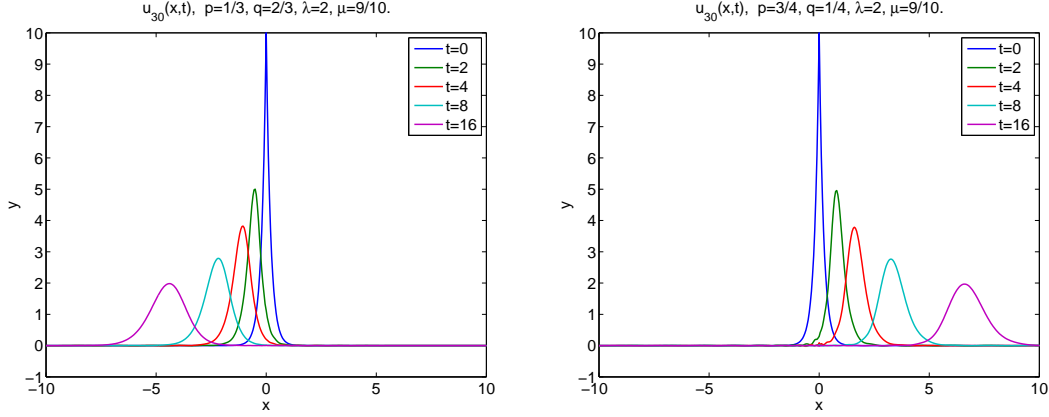


FIGURE 5.2. Left:  $p = 1/3$ ,  $q = 2/3$ . Right:  $p = 3/4$ ,  $q = 1/4$ .

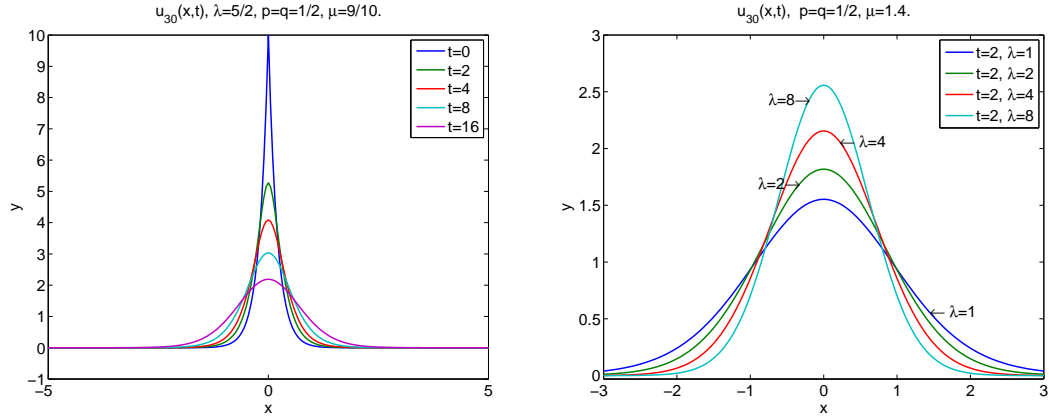
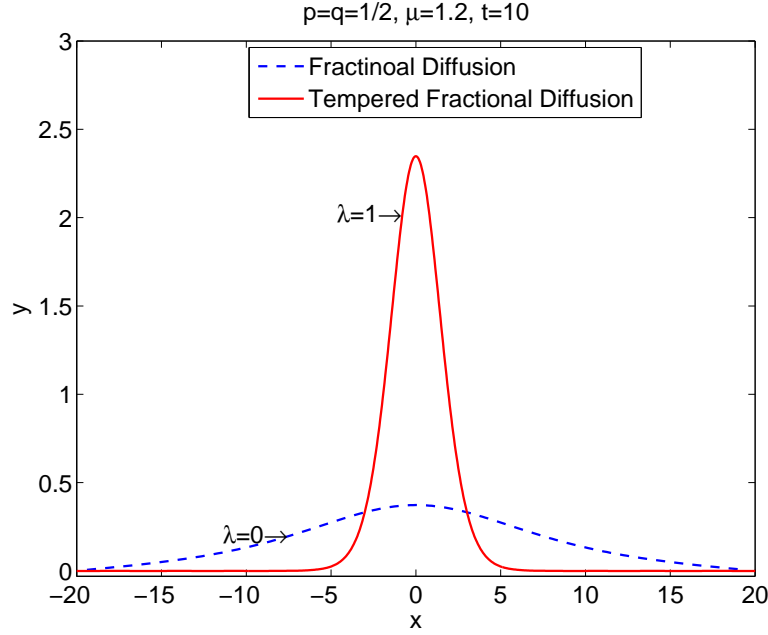


FIGURE 5.3. Left:  $p = q = 1/2$ ,  $\lambda = 5/2$ . Right:  $p = q = 1/2$ ,  $t = 2$ .

## 6. CONCLUDING REMARKS

We presented in this paper efficient spectral methods using the generalized Laguerre functions for solving the tempered fractional differential equations. Our numerical methods and analysis are based on an important observation that the tempered fractional derivative, when restricted to the half line, is intrinsically related to the generalized Laguerre functions that we defined in Sections 3. By exploring the properties of generalized Laguerre functions, we derived optimal approximation results in properly weighted Sobolev spaces, and showed that

we define two classes of generalized Laguerre functions, study their approximation properties, and apply them for solving simple one sided tempered fractional equations. In Section 4, we develop a spectral-Galerkin method for solving a tempered fractional diffusion equation on the half line. Finally, we present a spectral-Galerkin method for solving the tempered fractional diffusion equation on the whole line

FIGURE 5.4. Initial distribution  $u_0(x) = 10e^{-4x^2}$ .

## APPENDIX A. PROOF OF LEMMA 2.2

We first prove (2.43)-(2.44). Recall the fractional integral formula of hypergeometric functions (see [2, P. 287]): for real  $b, \mu \geq 0$ ,

$$x^{b+\mu-1} {}_1F_1(a; b+\mu; x) = \frac{\Gamma(b+\mu)}{\Gamma(b)\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} t^{b-1} {}_1F_1(a; b; t) dt, \quad x \in \mathbb{R}^+. \quad (\text{A.1})$$

Taking  $a = -n$ ,  $b = \alpha + 1$  and using the hypergeometric representation (2.35) of the Laguerre polynomials, we obtain

$$x^{\alpha+\mu} L_n^{(\alpha+\mu)}(x) = \frac{\Gamma(n+\alpha+\mu+1)}{\Gamma(n+\alpha+1)\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} t^\alpha L_n^{(\alpha)}(t) dt,$$

which yields (2.43), i.e.,

$${}_0I_x^\mu \{x^\alpha L_n^{(\alpha)}(x)\} = h_n^{\alpha, -\mu} x^{\alpha+\mu} L_n^{(\alpha+\mu)}(x).$$

Then, performing  ${}_0D_x^\mu$  on both sides and taking  $\alpha + \mu \rightarrow \alpha$ , we derive from the relation (2.7) that for  $\alpha - \mu > -1$ ,

$${}_0D_x^\mu \{x^\alpha L_n^{(\alpha)}(x)\} = \frac{1}{h_n^{\alpha-\mu, -\mu}} x^{\alpha-\mu} L_n^{(\alpha-\mu)}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-\mu+1)} x^{\alpha-\mu} L_n^{(\alpha-\mu)}(x).$$

This leads to (2.44).

We now turn to (2.45)-(2.46). According to [20, (6.146), P. 191] (or [22, (B-7.2), P. 307]), we have

$${}_xI_\infty^\mu \{e^{-x} L_n^{(\alpha+\mu)}(x)\} = e^{-x} L_n^{(\alpha)}(x), \quad \alpha > -1, \mu > 0.$$

Similarly, from the property:  ${}_xD_\infty^\mu {}_xI_\infty^\mu u(x) = u(x)$ , we derive

$${}_xD_\infty^\mu \{e^{-x} L_n^{(\alpha)}(x)\} = e^{-x} L_n^{(\alpha+\mu)}(x).$$

Finally, we prove (2.47). Noting that

$${}_1F_1(a; c; x) = e^x {}_1F_1(c - a; c; -x), \quad (\text{A.2})$$

(cf. [3, P. 191]), we derive from (2.35) and (A.2) that

$$\begin{aligned} x^\alpha L_n^{(\alpha)}(x) e^{-x} &= x^\alpha \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x) e^{-x} \\ &= \frac{(\alpha+1)_n}{n!} x^\alpha {}_1F_1(n+\alpha+1; \alpha+1; -x) = \frac{(\alpha+1)_n}{n!} x^\alpha {}_1F_1(n+\alpha+1; \alpha+1; -x) \quad (\text{A.3}) \\ &= \frac{(\alpha+1)_n}{n!} \sum_{j=0}^{\infty} \frac{(n+\alpha+1)_j (-1)^j}{(\alpha+1)_j} \frac{x^{j+\alpha}}{j!}. \end{aligned}$$

Then acting the derivative  $D^k$  on (A.3) and using the identities (2.35), (A.2) again, we obtain

$$\begin{aligned} D^k \{x^\alpha L_n^{(\alpha)}(x) e^{-x}\} &= \frac{(\alpha+1)_n}{n!} \sum_{j=0}^{\infty} \frac{(n+\alpha+1)_j (-1)^j}{(\alpha+1)_j} \frac{D^k x^{j+\alpha}}{j!} \\ &= \frac{(\alpha+1)_n}{n!} \sum_{j=0}^{\infty} \frac{(n+\alpha+1)_j (-1)^j}{(\alpha+1)_j} \frac{\Gamma(j+\alpha+1)}{\Gamma(j+\alpha-k+1)} \frac{x^{j+\alpha-k}}{j!} \\ &= x^{\alpha-k} \frac{(\alpha+1)_n}{n!} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)} \sum_{j=0}^{\infty} \frac{(n+\alpha+1)_j}{(\alpha-k+1)_j} \frac{(-x)^j}{j!} \\ &= x^{\alpha-k} \frac{(\alpha-k+1)_{n+k}}{n!} {}_1F_1(n+\alpha+1; \alpha-k+1; -x) \\ &= \frac{(n+k)!}{n!} x^{\alpha-k} \frac{(\alpha-k+1)_{n+k}}{(n+k)!} {}_1F_1(-n-k; \alpha-k+1; x) e^{-x} \\ &= \frac{\Gamma(n+k+1)}{\Gamma(n+1)} x^{\alpha-k} L_{n+k}^{(\alpha-k)}(x) e^{-x}. \end{aligned}$$

This ends the proof.

## APPENDIX B. THE PROOF OF (5.10) AND THE DETAIL ON THE ENTRIES OF **A**

### Proof of (5.10)

- for  $x \in \mathbb{R}^-$ ,  $0 < s < 1$ ,

$$\begin{aligned} -\infty D_x^{s,\lambda} \phi^*(x) &= -\infty I_x^{1-s,\lambda} -\infty D_x^{1,\lambda} \phi^*(x) = \frac{e^{-\lambda x}}{\Gamma(1-s)} \int_{-\infty}^x \frac{e^{\lambda \tau} (2\lambda) e^{\lambda \tau}}{(x-\tau)^s} d\tau \\ &\stackrel{t=x-\tau}{=} \frac{2\lambda e^{-\lambda x}}{\Gamma(1-s)} \int_0^\infty \frac{e^{2\lambda(x-t)}}{t^s} dt = \frac{(2\lambda)^s e^{\lambda x}}{\Gamma(1-s)} \int_0^\infty e^{-2\lambda t} (2\lambda t)^{-s} d(2\lambda t) \\ &= (2\lambda)^s e^{\lambda x}, \quad (\text{B.1}) \\ -\infty D_x^{s,\lambda} \phi_{n_1}^-(x) &= -\infty I_x^{1-s,\lambda} -\infty D_x^{1,\lambda} \phi_{n_1}^-(x) \stackrel{(3.10)}{=} -\infty I_x^{1-s,\lambda} \{-(n_1+1) \mathcal{L}_{n_1+1}^{(0,\lambda)}(-x)\} \\ &\stackrel{(2.45)}{=} -(n_1+1) (2\lambda)^{s-1} L_{n_1+1}^{(s-1)}(-2\lambda x) e^{-\lambda x} \\ -\infty D_x^{s,\lambda} \phi_{n_2}^+(x) &= 0 \end{aligned}$$

- for  $x \in \mathbb{R}^+$ ,  $0 < s < 1$ ,

$$\begin{aligned}
-_{\infty}D_x^{s,\lambda}\phi^*(x) &= -_{\infty}I_x^{1-s,\lambda} -_{\infty}D_x^{1,\lambda}\phi^*(x) = \frac{e^{-\lambda x}}{\Gamma(1-s)} \int_{-\infty}^0 \frac{2\lambda e^{2\lambda\tau}}{(x-\tau)^s} d\tau \stackrel{\tau=x-t}{=} \frac{2\lambda e^{\lambda x}}{\Gamma(1-s)} \int_x^{\infty} \frac{e^{-2\lambda t}}{t^s} dt, \\
-_{\infty}D_x^{s,\lambda}\phi_{n_1}^-(x) &= -_{\infty}I_x^{1-s,\lambda} -_{\infty}D_x^{1,\lambda}\phi_{n_1}^-(x) \stackrel{(3.10)}{=} \frac{e^{-\lambda x}}{\Gamma(1-s)} \int_{-\infty}^0 \frac{-(n_1+1)L_{n_1+1}^{(0)}(-2\lambda\tau)e^{2\lambda\tau}}{(x-\tau)^s} d\tau \\
&\stackrel{\tau=x-t}{=} -e^{\lambda x} \frac{n_1+1}{\Gamma(1-s)} \int_x^{\infty} \frac{L_{n_1+1}^{(0)}(2\lambda(t-x))e^{-2\lambda t}}{t^s} dt, \\
-_{\infty}D_x^{s,\lambda}\phi_{n_2}^+(x) &= -_{\infty}I_x^{1-s,\lambda} -_{\infty}D_x^{1,\lambda}\phi_{n_2}^+(x) \stackrel{(3.9)}{=} \frac{e^{-\lambda x}}{\Gamma(1-s)} \int_0^x \frac{(n_2+1)L_{n_2}^{(0)}(x)}{(x-\tau)^s} d\tau \\
&= \frac{\Gamma(n_2+2)}{\Gamma(n_2+2-s)} x^{1-s} \mathcal{L}_{n_2}^{(1-s,\lambda)}(x).
\end{aligned} \tag{B.2}$$

**The entries of matrix  $\mathbf{A}$  with  $1 < \mu = 1 + s < 2$ .**

$$\begin{aligned}
(-_{\infty}D_x^{s,\lambda}\phi^*, {}_x D_{\infty}^{1,\lambda}\phi^*) &= \frac{-(2\lambda)^2}{\Gamma(1-s)} \int_0^{\infty} \int_x^{\infty} \frac{e^{-2\lambda t}}{t^s} dt dx = \frac{-(2\lambda)^2}{\Gamma(1-s)} \int_0^{\infty} \frac{e^{-2\lambda t}}{t^s} \int_0^t 1 dx dt \\
&= \frac{-(2\lambda)^2}{\Gamma(1-s)} \int_0^{\infty} t^{1-s} e^{-2\lambda t} dt \stackrel{\tau=2\lambda t}{=} \frac{-(2\lambda)^s}{\Gamma(1-s)} \int_0^{\infty} \tau^{1-s} e^{-\tau} d\tau = (s-1)(2\lambda)^s.
\end{aligned} \tag{B.3}$$

Since

$$D\{(2\lambda x)L_{n_2+1}^{(1)}(2\lambda x)\} = 2\lambda(n_2+2)L_{n_2+1}^{(0)}(2\lambda x), \text{ i.e. } \int_0^t L_{n_2+1}^{(0)}(2\lambda x) dx = \frac{1}{n_2+2} t L_{n_2+1}^{(1)}(2\lambda t),$$

then,

$$\begin{aligned}
(-_{\infty}D_x^{s,\lambda}\phi^*, {}_x D_{\infty}^{1,\lambda}\phi_{n_2}^+) &= \frac{-2\lambda(n_2+1)}{\Gamma(1-s)} \int_0^{\infty} \int_x^{\infty} \frac{e^{-2\lambda t}}{t^s} dt L_{n_2+1}^{(0)}(2\lambda x) dx \\
&= \frac{-2\lambda(n_2+1)}{\Gamma(1-s)} \int_0^{\infty} \frac{e^{-2\lambda t}}{t^s} \int_0^t L_{n_2+1}^{(0)}(2\lambda x) dx dt \\
&= \frac{-2\lambda(n_2+1)}{(n_2+2)\Gamma(1-s)} \int_0^{\infty} t^{1-s} L_{n_2+1}^{(1)}(2\lambda t) e^{-2\lambda t} dt.
\end{aligned} \tag{B.4}$$

Similarly, we have

$$\begin{aligned}
(-_{\infty}D_x^{s,\lambda}\phi_{n_1}^-, {}_x D_{\infty}^{1,\lambda}\phi_{n_2}^+) &= \frac{(n_1+1)(n_2+1)}{\Gamma(1-s)} \int_0^{\infty} \int_x^{\infty} \frac{L_{n_1+1}^{(0)}(2\lambda(t-x))e^{-2\lambda t}}{t^s} dt L_{n_2+1}^{(0)}(2\lambda x) dx \\
&= \frac{(n_1+1)(n_2+1)}{\Gamma(1-s)} \int_0^{\infty} t^{-s} e^{-2\lambda t} \int_0^t L_{n_2+1}^{(0)}(2\lambda x) L_{n_1+1}^{(0)}(2\lambda(t-x)) dx dt \\
&\stackrel{x=t\xi}{=} \frac{(n_1+1)(n_2+1)}{\Gamma(1-s)} \int_0^{\infty} t^{1-s} e^{-2\lambda t} \int_0^1 L_{n_2+1}^{(0)}(2\lambda t\xi) L_{n_1+1}^{(0)}(2\lambda t(1-\xi)) d\xi dt.
\end{aligned} \tag{B.5}$$

**The entries of matrix  $\mathbf{A}$  with  $0 < \mu = s < 1$ .**

$$\begin{aligned}
(-_{\infty}D_x^{s,\lambda}\phi^*, \phi^*) &= (2\lambda)^s \int_{-\infty}^0 e^{2\lambda x} dx + \frac{2\lambda}{\Gamma(1-s)} \int_0^{\infty} \int_x^{\infty} \frac{e^{-2\lambda t}}{t^s} dt dx \\
&= (2\lambda)^{s-1} + \frac{2\lambda}{\Gamma(1-s)} \int_0^{\infty} \frac{e^{-2\lambda t}}{t^s} \int_0^t 1 dx dt = (2\lambda)^{s-1} + \frac{2\lambda}{\Gamma(1-s)} \int_0^{\infty} t^{1-s} e^{-2\lambda t} dt \\
&\stackrel{\tau=2\lambda t}{=} (2\lambda)^{s-1} + \frac{(2\lambda)^{s-1}}{\Gamma(1-s)} \int_0^{\infty} \tau^{1-s} e^{-\tau} d\tau = (2-s)(2\lambda)^{s-1}.
\end{aligned} \tag{B.6}$$

Owe to

$$D\{(2\lambda x)^2 L_{n_2}^{(2)}(2\lambda x)\} = (2\lambda)^2(n_2 + 2)xL_{n_2}^{(1)}(2\lambda x),$$

i.e.,

$$\int_0^t xL_{n_2}^{(1)}(2\lambda x)dx = \frac{1}{n_2 + 2}t^2L_{n_2}^{(2)}(2\lambda t),$$

we obtain that

$$\begin{aligned} (-\infty D_x^{s,\lambda} \phi^*, \phi_{n_2}^+) &= \int_0^\infty \frac{e^{-\lambda x}}{\Gamma(1-s)} \int_{-\infty}^0 \frac{2\lambda e^{2\lambda\tau}}{(x-\tau)^s} d\tau xL_{n_2}^{(1)}(2\lambda x)e^{-\lambda x} dx \\ &\stackrel{\tau=x-t}{=} \frac{2\lambda}{\Gamma(1-s)} \int_0^\infty \int_x^\infty \frac{e^{-2\lambda t}}{t^s} dt xL_{n_2}^{(1)}(2\lambda x) dx = \frac{2\lambda}{\Gamma(1-s)} \int_0^\infty \frac{e^{-2\lambda t}}{t^s} \int_0^t xL_{n_2}^{(1)}(2\lambda x) dx dt \quad (\text{B.7}) \\ &= \frac{2\lambda}{(n_2 + 2)\Gamma(1-s)} \int_0^\infty t^{2-s} L_{n_2}^{(2)}(2\lambda t) e^{-2\lambda t} dt. \end{aligned}$$

Similarly, we have

$$\begin{aligned} (-\infty D_x^{s,\lambda} \phi_{n_1}^-, \phi_{n_2}^+) &= -\frac{n_1 + 1}{\Gamma(1-s)} \int_0^\infty \int_x^\infty \frac{L_{n_1+1}^{(0)}(2\lambda(t-x))e^{-2\lambda t}}{t^s} dt xL_{n_2}^{(1)}(2\lambda x) dx \\ &= -\frac{n_1 + 1}{\Gamma(1-s)} \int_0^\infty t^{-s} e^{-2\lambda t} \int_0^t xL_{n_2}^{(1)}(2\lambda x) L_{n_1+1}^{(0)}(2\lambda(t-x)) dx dt \quad (\text{B.8}) \\ &\stackrel{x=t\xi}{=} -\frac{n_1 + 1}{\Gamma(1-s)} \int_0^\infty t^{2-s} e^{-2\lambda t} \int_0^1 \xi L_{n_2}^{(1)}(2\lambda t\xi) L_{n_1+1}^{(0)}(2\lambda t(1-\xi)) d\xi dt \end{aligned}$$

The above equations are enough to calculate out the matrix **A** due to some symmetric properties of the entries.

## REFERENCES

- [1] M. Abramovitz and I.A. Stegun. *Handbook of Mathematical Functions*. Dover, New York, 1972.
- [2] G.E. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.
- [3] George E Andrews, Richard Askey, Ranjan Roy, et al. *Special functions*, encyclopedia of mathematics and its applications, 1999.
- [4] B. Baeumer, D.A. Benson, M.M. Meerschaert, and S. W. Wheatcraft. Subordinated advection-dispersion equation for contaminant transport. *Water Resources Research*, 37(6):1543–1550, 2001.
- [5] B. Baeumer, M. Kovács, and M.M. Meerschaert. Fractional reproduction-dispersal equations and heavy tail dispersal kernels. *Bulletin of Mathematical Biology*, 69(7):2281–2297, 2007.
- [6] Peter Carr, Hélyette Geman, Dilip B Madan, and Marc Yor. The fine structure of asset returns: An empirical investigation\*. *The Journal of Business*, 75(2):305–333, 2002.
- [7] Sheng Chen, Jie Shen, and Li-Lian Wang. Generalized Jacobi functions and their applications to fractional differential equations. *Math. Comp.*, 85(300):1603–1638, 2016.
- [8] J.H. Cushman and T.R. Ginn. Fractional advection-dispersion equation: a classical mass balance with convolution-fickian flux. *Water resources research*, 36(12):3763–3766, 2000.
- [9] Z.Q. Deng, L. Bengtsson, and V.P. Singh. Parameter estimation for fractional dispersion model for rivers. *Environmental Fluid Mechanics*, 6(5):451–475, 2006.
- [10] Kai Diethelm. *The Analysis of Fractional Differential Equations, Lecture Notes in Math., Vol. 2004*. Springer, Berlin, 2010.
- [11] R. Gorenflo, F. Mainardi, E. Scalas, and M. Raberto. Fractional calculus and continuous-time finance iii: the diffusion limit. In *Mathematical finance*, pages 171–180. Springer, 2001.
- [12] Can Huang, Qingshuo Song, and Zhimin Zhang. Spectral collocation method for substantial fractional differential equations. *Submitted*.
- [13] J.H. Jeon, H.M.S. Monne, M. Javanainen, and R. Metzler. Anomalous diffusion of phospholipids and cholesterol in a lipid bilayer and its origins. *Physical review letters*, 109(18):188103, 2012.
- [14] M. Magdziarz, A. Weron, and K. Weron. Fractional fokker-planck dynamics: Stochastic representation and computer simulation. *Physical Review E*, 75(1):016708, 2007.
- [15] Mark M Meerschaert, Yong Zhang, and Boris Baeumer. Tempered anomalous diffusion in heterogeneous systems. *Geophysical Research Letters*, 35(17), 2008.

- [16] M.M. Meerschaert and E. Scalas. Coupled continuous time random walks in finance. *Physica A: Statistical Mechanics and its Applications*, 370(1):114–118, 2006.
- [17] R. Metzler and J. Klafter. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, 339(1):1–77, 2000.
- [18] R. Metzler and J. Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *Journal of Physics A: Mathematical and General*, 37(31):R161, 2004.
- [19] A. Piryatinska, A. Saichev, and W. Woyczynski. Models of anomalous diffusion: the subdiffusive case. *Physica A: Statistical Mechanics and its Applications*, 349(3):375–420, 2005.
- [20] I. Podlubny. *Fractional Differential Equations*, volume 198 of *Mathematics in Science and Engineering*. Academic Press Inc., San Diego, CA, 1999. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.
- [21] I. Podlubny. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Academic press, 1999.
- [22] G. Y. Popov. Concentration of elastic stresses near punches, cuts, thin inclusions and supports. *Nauka, Moscow*, 1:982, 1982.
- [23] F. Sabzikar, M.M. Meerschaert, and J. Chen. Tempered fractional calculus. *Journal of Computational Physics*, 2014.
- [24] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional integrals and derivatives*. Gordon and Breach Science Publ., 1993.
- [25] E. Scalas. Five years of continuous-time random walks in econophysics. In *The complex networks of economic interactions*, pages 3–16. Springer, 2006.
- [26] J. Shen, T. Tang, and L.L. Wang. *Spectral Methods: Algorithms, Analysis and Applications*, volume 41 of *Series in Computational Mathematics*. Springer-Verlag, Berlin, Heidelberg, 2011.
- [27] E.M. Stein and G.L. Weiss. *Introduction to Fourier analysis on Euclidean spaces*, volume 1. Princeton university press, 1971.
- [28] G. Szegő. *Orthogonal Polynomials (Fourth Edition)*. AMS Coll. Publ., 1975.
- [29] Mohsen Zayernouri, Mark Ainsworth, and George Em Karniadakis. Tempered fractional Sturm-Liouville eigenproblems. *SIAM J. Sci. Comput.*, 37(4):A1777–A1800, 2015.
- [30] C. Zhang and B.Y. Guo. Domain decomposition spectral method for mixed inhomogeneous boundary value problems of high order differential equations on unbounded domains. *Journal of Scientific Computing*, 53(2):451–480, 2012.
- [31] Yong Zhang and Mark M Meerschaert. Gaussian setting time for solute transport in fluvial systems. *Water Resources Research*, 47(8), 2011.